Super Yang-Mills theories and the structure of anomalies and spontaneous parameters

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Seminar at the TU in Vienna
November 2003

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Topics

1 Logical possibilities in super Yang-Mills theories

2 supercurrent algebra

3 on the road of conserved susy

4 the predominant role of
$F (x) = \frac{1}{4} \left(F_{\mu\nu}^{A} F^{\mu\nu \ A} \right) (x)$
and $\mathcal{B}^{2} = \langle \Omega \mid F (x) \mid \Omega \rangle$

5 susy breaking : through anomalies and spontaneously $\rightarrow$ **mass gap**

6 conclusions - outlook

Contents

1 Logical possibilities in super Yang-Mills theories

$$\langle \Omega | \vartheta_{\mu\nu} | \Omega \rangle = \varepsilon \, g_{\mu\nu} \; ; \; g_{\mu\nu} = diag(1, -1 \otimes 3)$$

$$\varepsilon = \begin{cases} > 0 \leftrightarrow + \\ = 0 \leftrightarrow 0 \\ < 0 \leftrightarrow - \end{cases}$$

original focus was on cases b_0, b_+

cases	susy breaking: anomalous current divergences $j_{\mu\alpha}$, $j^*_{\nu\dot\beta}$	susy breaking: spontaneous	massless goldstino	ε
a_-	yes	yes	no	-
(to be) excluded				
a_0	yes	no	no	0
a_+	yes	yes	no	+
b_+	no	yes	yes	+
b_-	no	yes	yes	-
b_0	no	no	no	0

Wien2003

2 Supercurrent algebra [1]

supercharges $Q_{\alpha}\ ,\ Q^{*}_{\dot{\beta}}$

supercurrents $j_{\mu\,\alpha}\ ,\ j^{*}_{\nu\,\dot{\beta}}$

we follow the path b_{+} $\leftrightarrow$ **susy broken *only* spontaneously**

$$Q_{\alpha} = \int_{t} d^{3}x\, j_{0\,\alpha}(t, x)\ ;\ Q_{\alpha} \to Q^{*}_{\dot{\alpha}}$$

$$P_{\mu} = \int_{t} d^{3}x\, \vartheta_{0\,\mu}(t, x)$$

the 'once local' form of the susy relation is

$$\begin{aligned} &\int d^{3}x \left\{ j_{0\,\alpha}(t, x), j^{*}_{0\,\dot{\beta}}(t, y) \right\} \\ &= \int d^{3}x\, \sigma^{\mu}_{\alpha\dot{\beta}}\, \vartheta_{0\,\mu}(t, y)\, \delta^{3}(\vec{x} - \vec{y}) \end{aligned} \tag{1}$$

[1] L. Bergamin and P. Minkowski, *Spontaneous Susy Breaking in N=2 Super-Yang-Mills Theories*, Contribution to 30th International Conference on High-Energy Physics (ICHEP 2000), Osaka, Japan, 27 Jul - 2 Aug 2000, Published in *Osaka 2000, High energy physics, vol. 2* 1387-1388, hep-ph/0011041.

the relation in eq. (1) can be written in covariant form, using general space-time positions $x = (t , \vec{x})$ and $x \to y$

$$\int d^{4}x \, e^{i q x} \, \partial_{x}^{\mu} \, T \left(j_{\mu \alpha} (x) \, j^{*}_{\nu \dot{\beta}} (y) \right)$$

$$= \sigma^{\varrho}_{\alpha \dot{\beta}} \, \vartheta_{\nu \varrho} (y) \quad \textbf{for } \nu = 0 \tag{2}$$

In eq. (2) T denotes time ordereing. Irrespective of eventual Schwinger terms the once integrated relation (2) can be extended to all values of ν. Introducing the Fourier transform of the time ordered susy current correlation function eq. (2) becomes

$$\tau_{\mu\alpha \, ; \, \nu\dot{\beta}} (q) = \int d^{4}x \, e^{i q x} \times$$

$$\times \langle \Omega \mid T \left(j_{\mu \alpha} (x) \, j^{*}_{\nu \dot{\beta}} (0) \right) \mid \Omega \rangle$$

$$\frac{1}{i} \, q^{\mu} \, \tau_{\mu\alpha \, ; \, \nu\dot{\beta}} (q) = \sigma^{\varrho}_{\alpha \dot{\beta}} \, \langle \Omega \mid \vartheta_{\nu \varrho} (0) \mid \Omega \rangle \tag{3}$$

Eqs. (2) , (3) are precisely valid if the susy currents are conserved $\partial^{\mu} j_{\mu\,\alpha}(x) = 0$.

We can evaluate eq. (3) for $q \rightarrow 0$ using the Lorentz covariant decomposition

$$
\begin{aligned}
\tau_{\mu\alpha\,;\,\nu\dot{\beta}}(q) \;=\;& \Gamma_{\mu\nu\varrho}(q)\,\sigma^{\varrho}_{\alpha\dot{\beta}}\,\tau(q^{2}) + \\
& + \textbf{transverse terms} \\
\Gamma_{\mu\nu\varrho}(q) \;=\;& g_{\mu\varrho}\, q_{\nu} + g_{\nu\varrho}\, q_{\mu} - g_{\mu\nu}\, q_{\varrho} \\
q^{\mu}\,\Gamma_{\mu\nu\varrho}(q) \;=\;& q^{2}\, g_{\nu\varrho}
\end{aligned}
\tag{4}
$$

and obtain

$$
\begin{aligned}
&\textbf{for } \langle\,\Omega\,|\,\vartheta_{\nu\varrho}(0)\,|\,\Omega\,\rangle = \varepsilon\, g_{\nu\varrho} \\
&\lim_{q \rightarrow 0} q^{2}\,\tau(q^{2}) = \varepsilon
\end{aligned}
\tag{5}
$$

From eqs. (4) , (5) we deduce that for $\varepsilon \neq 0$, τ must inherit a goldstino induced

pole of the form

$$\Gamma_{\mu\nu\varrho}(q)\,\sigma^{\varrho}_{\alpha\dot\beta}\,Abs\,\tau(q^2) =$$

$$= \int d\varrho(m^2) \sum_n \begin{pmatrix} \langle \Omega \mid j_{\mu\alpha}(0) \mid p; n \rangle \times \\ \times \langle p; n \mid j^{*}_{\nu\dot\beta}(0) \mid \Omega \rangle \end{pmatrix}$$

$$\varrho(m^2) > 0$$

$$\downarrow$$

$$q \to 0 : \tau \sim \mid f_g \mid^2 / (q^2 + i\eta) \tag{6}$$

In eq. (6) the positive measure $\varrho(m^2)$ refers to the Kallen-Lehmann representation for the two point function $\tau_{\mu\alpha\,;\,\nu\dot\beta}(q)$ (4) and f_g denotes the goldstino 'decay constant' analogous to f_π for pions, but of dimension $mass^2$, which is defined modulo an arbitrary phase.

$\eta \downarrow +0$ is used for the infinitesimal

positive imaginary part of q^2 to distinguish it from the vacuum energy density ε .

Combining eqs. (5) and (6) we obtain

$$
\begin{aligned}
& | f_g |^2 = \varepsilon \geq 0 \\
& \langle \Omega | j_{\mu\alpha}(0) | p ; s \, goldstino \rangle = \\
& = f_g \sigma_{\mu \, \alpha\dot{\beta}} u^{\dot{\beta}} (p ; s)
\end{aligned} \tag{7}
$$

Eq. (7) eliminates the case b_- in the table of section 1.

3 On the road of conserved susy

In this section we follow, for N=1 super Yang-Mills theory with (simple) gauge group G, the road b_0 of the table in section 1, i.e. $\varepsilon = 0$, as long as possible, ignoring the prejudice originating from the trace anomaly :

$$\begin{aligned} \vartheta^{\mu}_{\mu} &= (-2\beta(g)/g)\,\mathcal{L} \\ &= -(2\beta_0)\left[\tfrac{1}{4}F^{A}_{\mu\nu}F^{A\,\mu\nu}\right]_{ren.gr.inv.} \\ \beta_0 &= -b_0 = 3\,C_2(G)/(16\pi^2) \\ &\to \varepsilon < 0 \end{aligned} \tag{8}$$

In eq. (8) the suffix $_{ren.gr.inv.}$ – dropped subsequently – denotes the field strength bilinear operator renormalized in a renormalization group invariant way. This presents no (obvious) problems for $\beta_0 > 0$.

$C_2(G)$ denotes the second Casimir operator of the gauge group, with the normalization $C_2(SU_n) = n$.

The question we ask is : how does the (N=1) susy covariant effective action for the composite operators of the Lagrangean multiplet determine the vacuum - or spontaneous parameters

$$\begin{aligned} &\langle \Omega \mid \vartheta_{\mu\nu} \mid \Omega \rangle = \varepsilon \, g_{\mu\nu} \\ &\langle \Omega \mid \Lambda^{A\alpha} \Lambda^{A}_{\alpha} \mid \Omega \rangle \end{aligned} \tag{9}$$

In eq. (9) $\Lambda^{A\alpha}(x)$ denote the gaugino fields normalized in accord with the field strengths $F^{A}_{\mu\nu}(x)$ [2] .

[2] some selected references :

G. Veneziano and S. Yankielowicz, Phys. Lett. B113 (1982) 231,

G. M. Shore, Nucl. Phys. B222 (1983) 446,

R. Dijkgraaf and C. Vafa, hep-th/0208048,

R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, Phys.Lett. B573 (2003) 138, hep-th/0211017,

L. Bergamin and P. Minkowski, hep-th/0301155.

We called this effective action

'the minimal source extension'

$\downarrow$

a) chiral (base) superfield

First let me discuss a primary (base) chiral superfield Φ subjected to the constraint $\overline{D}_{\dot{\beta}} \, \Phi = 0$

$$\Phi = \begin{cases} \quad \vartheta^{2} H (x^{-}) & +2 \\ + \vartheta^{\alpha} \eta_{\alpha} (x^{-}) & -2 \\ + \varphi (x^{-}) & +2 \end{cases} \tag{10}$$

$$x^{-\mu} = x^{\mu} - \frac{i}{2} \vartheta^{\alpha} \sigma^{\mu}_{\alpha\dot{\beta}} \overline{\vartheta}^{\dot{\beta}}$$

The signed number in the last column of the bracket in eq. (10) denotes the bosonic (+) and fermionic (-) number of degrees of freedom per unit phase space pertaining to the given local fields H, η_{α}, φ.

The complex scalar field H (for 'Hilfsfeld') is thought to be an auxiliary field, fully determined from the dynamical variables

$$H \;=\; H\,(\,\eta_{\,\alpha}\,,\;\varphi\,) \tag{11}$$

so that susy is to be achieved ignoring the highest $\vartheta^{\,2}$ component in eq. (10) .

Expanding $x^{\,-}$ we obtain the full chiral scalar field

$$\Phi \;=$$

$$\left\{ \begin{array}{c} \vartheta^{\,2}\,\overline{\vartheta}^{\,2}\,\left(\,-\,\frac{1}{4}\,\Box\,\varphi\,\right) \\ \vartheta^{\,2}\,\overline{\vartheta}_{\,\dot{\delta}}\,\left(\,\frac{-i}{2}\,\partial^{\,\dot{\delta}\alpha}\,\eta_{\,\alpha}\,\right)\,+\,\overline{\vartheta}^{\,2}\,\vartheta^{\,\alpha}\,0 \\ \vartheta^{\,2}\,H\,+\,\vartheta^{\,\alpha}\,\sigma^{\,\mu}_{\alpha\dot{\beta}}\,\overline{\vartheta}^{\,\dot{\beta}}\,\left(\,\frac{-i}{2}\,\partial_{\,\mu}\,\varphi\,\right)\,+\,\overline{\vartheta}^{\,2}\,0 \\ +\,\overline{\vartheta}^{\,\dot{\beta}}\,0\,+\,\vartheta^{\,\alpha}\,\eta_{\,\alpha} \\ +\,\varphi \end{array} \right\} \tag{12}$$

For full transparency we exhibit the antichiral field $\overline{\Phi}$

$$\overline{\Phi} \;=$$

$$\left\{ \begin{array}{c} \vartheta^{\,2}\,\overline{\vartheta}^{\,2}\,\left(-\,\frac{1}{4}\,\Box\,\varphi^{\,*}\right) \\ \vartheta^{\,2}\,\overline{\vartheta}_{\,\dot{\delta}}\,0 \;+\; \overline{\vartheta}^{\,2}\,\vartheta^{\,\alpha}\,\left(\,\frac{i}{2}\,\partial^{\,\dot{\gamma}}_{\,\alpha}\,\eta^{\,*}_{\,\dot{\gamma}}\,\right) \\ \vartheta^{\,2}\,0 \;+\; \vartheta^{\,\alpha}\,\sigma^{\,\mu}_{\alpha\dot{\beta}}\,\overline{\vartheta}^{\,\dot{\beta}}\left(\,\frac{i}{2}\,\partial_{\,\mu}\,\varphi^{\,*}\right) \;+\; \overline{\vartheta}^{\,2}H^{\,*} \\ +\,\vartheta^{\,\alpha}\,0 \;+\; \overline{\vartheta}_{\,\dot{\gamma}}\,\eta^{\,*\,\dot{\gamma}} \\ +\,\varphi^{\,*} \end{array} \right\} \tag{13}$$

Supersymmetry in the sense of manifestly equal number of bosonic and fermionic degrees of freedom is clearly not obvious through the chiral constraint (eqs. 12 and 13) .

We note the structure of the highest component of $\overline{\Phi}\,\Phi$

$$\overline{\Phi}\,\Phi\,\Big|_{\vartheta^2\,\overline{\vartheta}^2} = \left\{\begin{array}{l} H^*\,H - \frac{1}{4}\,\Box\,(\varphi^*\,\varphi) \\ + (\partial^\mu\,\varphi^*)\,(\partial_\mu\,\varphi) \\ + \eta^*_{\dot\gamma}\,\frac{i}{2}\,\overleftrightarrow{\partial}_\mu\,\sigma^{\mu\,\dot\gamma\alpha}\,\eta_\alpha \end{array}\right\} \tag{14}$$

In eq. (14) we recognize the kinetic Lagrangean density for the complex scalar field φ and the irreducible (Majorana-) spinor η_α.

$\Box\,(\varphi^*\,\varphi)$ **is relevant for the construction of the energy momentum tensor for the scalar field φ, whereas the term $H^*\,H$ interpreted as a negative potential, is not bounded from below, signalling the special role of the 'auxiliary' field H.**

4 Grassmann variables and base for susy algebra

Grassmann variables ϑ , η , $\cdots$ are chosen in connection with $SL\ (2, C)$ representations for spin

$$
\begin{array}{l}
\textbf{right chiral} \ : \ \vartheta_{\alpha} \ , \ \eta_{\alpha} \ , \ \cdots \ ; \ \alpha \ = \ 1, 2 \\
\vartheta_{\alpha} \vartheta_{\beta} + \vartheta_{\beta} \vartheta_{\alpha} \ \equiv \ 0 \\
SL\ (2, C) \ : \ \vartheta_{\alpha} \ \rightarrow \ \mathcal{A}_{\alpha\beta} \vartheta_{\beta} \\
\textbf{left chiral} \ : \ \overline{\vartheta}^{\dot{\gamma}} \ , \ \overline{\eta}^{\dot{\gamma}} \ , \ \cdots \ ; \ \dot{\gamma} \ = \ 1, 2 \\
\overline{\vartheta}^{\dot{\gamma}} \overline{\vartheta}^{\dot{\delta}} + \overline{\vartheta}^{\dot{\delta}} \overline{\vartheta}^{\dot{\gamma}} \ \equiv \ 0 \\
SL\ (2, C) \ : \ \overline{\vartheta}^{\dot{\gamma}} \ \rightarrow \ \widetilde{\mathcal{A}}^{\dot{\gamma}\dot{\delta}} \overline{\vartheta}^{\dot{\delta}} \\
\widetilde{\mathcal{A}} \ = \ \left(\mathcal{A}^{\dagger} \right)^{-1} \ = \ \varepsilon \ \overline{\mathcal{A}} \ \varepsilon^{-1}
\end{array}
\tag{15}
$$

In eq. (15) ε denotes the symplectic invariant ($SL\ (2, C) \ \equiv \ SP\ (1, C)$)

$$
\left(\varepsilon \right)_{\alpha\beta} \ = \ \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)_{\alpha\beta} \ = \ \left(\overline{\varepsilon} \right)_{\dot{\alpha}\dot{\beta}}
\tag{16}
$$

with the inverse

$$
\left(\varepsilon^{\prime} \right)^{\alpha\beta} \ = \ - \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)_{\alpha\beta} \ = \ \left(\overline{\varepsilon}^{\prime} \right)^{\dot{\alpha}\dot{\beta}}
\tag{17}
$$

The pair ε , ε' can be used to lower and raise spinor indices respectively. Note however, that this process does not yield $\varepsilon \leftrightarrow \varepsilon'$ when applied to these invariants themselves, rather leaves them *invariant* **.**

The $SL(2,C)$ representations $\mathcal{A}$, $\tilde{\mathcal{A}}$ and equivalents

These form the 'quadrangle' represented by

$$
\begin{array}{ccc}
\theta_{\alpha}\ ;\ \mathcal{A} & \longleftrightarrow & \theta^{\alpha} = \left(\varepsilon'\right)^{\alpha\beta} \vartheta_{\beta}\ ;\ \mathcal{A}' \\
\updownarrow & & \updownarrow \\
\overline{\theta}_{\dot{\alpha}}\ ;\ \overline{\mathcal{A}} & \longleftrightarrow & \overline{\theta}^{\dot{\alpha}} = \left(\varepsilon'\right)^{\dot{\alpha}\dot{\beta}} \overline{\vartheta}_{\dot{\beta}}\ ;\ \overline{\mathcal{A}}'
\end{array}
\tag{18}
$$

$$
\mathcal{A}' = \varepsilon\, \mathcal{A}\, \varepsilon^{-1}\ ;\ \overline{\mathcal{A}}' = \varepsilon\, \overline{\mathcal{A}}\, \varepsilon^{-1} = \tilde{\mathcal{A}}
$$

The right chiral spinors ϑ_{α} , ϑ^{α} thus transform under the equivalent pair $\mathcal{A}$, $\mathcal{A}'$, whereas left chiral ones under the associated equivalent pair $\overline{\mathcal{A}}$, $\overline{\mathcal{A}}'$. But right and left chiral spinors transform inequivalently.

In the above 'quadrangular' characterization the Grassmann variables just represent any spinors, i.e. only the representations under $SL(2,C)$ are relevant.

4a Insertion on the $SL\,(2,C)$ matrices $\mathcal{A}\,,\cdots$

Connection to Lorentz transformations :

four vectors $v^{\mu}\,,\cdots$ are represented in chiral spinor coordinates as

$$v \;\rightarrow\; v_{\alpha\dot\beta} \;=\; v^{\mu}\,\sigma_{\mu\;\alpha\dot\beta} \;=\; \widetilde{v}$$

$$\widetilde{v} \;=\; \begin{pmatrix} v^{0} + v^{3} & v^{1} - i\,v^{2} \\ v^{1} + i\,v^{2} & v^{0} - v^{3} \end{pmatrix} \tag{19}$$

$$\widetilde{v} \;=\; v^{0}\,\sigma_{0} + \vec{v}\,\vec{\sigma} \;\;;\;\; v^{2} \;=\; Det\;\widetilde{v}$$

In eq. (19) $\sigma_{\mu} \;=\; (\,\sigma_{0}\,,\;\vec{\sigma}\,)$ denote the 2 by 2 unit matrix (σ_{0}) and the three Pauli matrices (σ_{k}) respectively.

Thus the transformation

$$\widetilde{v} \;\rightarrow\; \widetilde{w} \;=\; \mathcal{A}\,\widetilde{v}\,\mathcal{A}^{\dagger}$$

$$\rightarrow\; w \;=\; \Lambda\,v \;\textbf{or}\; w^{\mu} \;=\; \Lambda^{\mu}{}_{\nu}\,v^{\nu} \tag{20}$$

$$Det\;\Lambda \;=\; 1 \;\;;\;\; \Lambda^{0}{}_{0} \;\geq\; 1$$

Λ induces a (real) Lorentz transformation. The latter preserves the time ordering for causal v ($v^{2} \;\geq\; 0$) and has 4 by 4 determinant 1.

In fact the association $\mathcal{A} \rightarrow \Lambda(\mathcal{A})$ is 2 to 1

$$\Lambda(\mathcal{A}) = \Lambda(-\mathcal{A}) \rightarrow SO(1,3) \simeq SL(2,C) / Z_2 \tag{21}$$

maximal compact subgroups of $SO(1,3)$ and $SL(2,C)$:

The maximal compact subgroups are $SO3$ and $SU2$ respectively ($SO3 \simeq SU2 / Z_2$) .

Assuming the first ($SO3$) known, we look at rotation matrices, i.e. Lorentz transformations, which do not change the time-component and the association in eq. (20)

$$\Lambda = R \rightarrow w^0 = R^0{}_\mu v^\mu = v^0 \; \forall v \rightarrow$$
$$2w^0 = tr\, \mathcal{A} \tilde{v} \mathcal{A}^\dagger = tr\, \mathcal{A}^\dagger \mathcal{A} \tilde{v} = tr\, \tilde{v} \; \forall v$$
$$\rightarrow \mathcal{A}^\dagger \mathcal{A} = \P \text{ i.e. } \mathcal{A} = U \; ; \; ; \text{ qed} \tag{22}$$

[a]

Let us construct $\mathcal{A}_3$ associated with special Lorentz transformations along the 3-axis and U_3 associated with rotations around the 3-axis to conclude this section.

[a] Show the content of eq. (22) .

boost in the 3 direction

The $SO\,(1,3)$ transformation is, using lightcone coordinates

$$v^{\pm} = v^{0} \pm v^{3} : w^{\pm} = e^{\pm\chi} v^{\pm}$$

$$v_{\perp} = (v^{1}, v^{2}) : w_{\perp} = v_{\perp} \rightarrow$$

$$\widetilde{w} = \begin{pmatrix} e^{\chi} v^{+} & v^{1} - i v^{2} \\ v^{1} + i v^{2} & e^{-\chi} v^{-} \end{pmatrix} \tag{23}$$

$$\rightarrow \mathcal{A}_{3}(\chi) = \pm \begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix}$$

rotation around the 3 direction (right-screw convention)

The rotation around the 3 axis yields

$$w^{1} \pm i w^{2} = e^{\pm i\varphi} (v^{1} \pm i v^{2}) ; w^{\pm} = v^{\pm}$$

$$\widetilde{w} = \begin{pmatrix} v^{+} & e^{-i\varphi} (v^{1} - i v^{2}) \\ e^{+i\varphi} (v^{1} + i v^{2}) & v^{-} \end{pmatrix}$$

$$\rightarrow \mathcal{U}_{3}(\varphi) = \pm \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{+i\varphi/2} \end{pmatrix} \tag{24}$$

4b) Base susy algebra

We consider the fermionic operators $Q_{\alpha} \, , \, Q^{*}_{\dot\beta}$ **in conjunction with Grassmann variables** $\eta_{\alpha} \, , \, \overline{\eta}_{\dot\beta} \, ; \, \vartheta_{\alpha} \, , \, \overline{\vartheta}_{\dot\beta} \, ; \, \cdots$ **. The operators** $Q_{\alpha} \, , \, Q^{*}_{\dot\beta}$ **shall obey the anticommutation algebra**

$$\begin{aligned} & \left\{ Q_{\alpha} \, , \, Q^{*}_{\dot\beta} \right\} \; = \; P^{\mu} \, \sigma_{\mu \; \alpha \dot\beta} \\ & \{ \, Q_{\alpha} \, , \, Q_{\beta} \, \} \; = \; 0 \; ; \; \left\{ Q^{*}_{\dot\alpha} \, , \, Q^{*}_{\dot\beta} \right\} \; = \; 0 \end{aligned} \tag{25}$$

In eq. (25) P^{μ} **denote the components of the (self-adjoint) energy momentum four vector.**

The nontrivial anticommutation relation in eq. (25) can be 'bosonized' by means of Grassmann variables $\eta \, , \, \varepsilon$ **to become a commutation relation**

$$\begin{aligned} & \left[\, \eta \, Q \, , \, \overline{Q} \, \overline{\vartheta} \, \right] \; = \; P^{\mu} \, v_{\mu} \, (\, \eta \, , \, \overline{\vartheta} \,) \\ & \eta \, Q \; = \; \eta^{\alpha} \, Q_{\alpha} \; ; \; \overline{Q} \, \overline{\vartheta} \; = \; Q^{*}_{\dot\beta} \, \overline{\varepsilon}^{\dot\beta} \\ & v_{\mu} \, (\, \eta \, , \, \overline{\vartheta} \,) \; = \; \eta \, \sigma_{\mu} \, \overline{\vartheta} \; = \; \eta^{\alpha} \, \sigma_{\mu \; \alpha \dot\beta} \, \overline{\vartheta}^{\dot\beta} \end{aligned} \tag{26}$$

So we consider the action of the 'unitary' aperators

$$U \, (\, \eta \,) \; = \; \exp \left[\, \frac{1}{i} \, \left(\, \eta \, Q \; + \; \overline{Q} \, \overline{\eta} \, \right) \, \right] \tag{27}$$

on the substrate formed by

$$F(\vartheta, \overline{\vartheta}, x) = \exp\left[\frac{1}{i}\left(\vartheta Q + \overline{Q}\,\overline{\vartheta}\right)\right] e^{i x^{\mu} P_{\mu}} \tag{28}$$

by left-multiplication

$$\begin{aligned} F_U(\vartheta, \overline{\vartheta}, x) &= U(\eta) F(\vartheta, \overline{\vartheta}, x) \\ &= F(\vartheta + \eta, \overline{\vartheta} + \overline{\eta}, x + y) \\ y^{\mu} &= \frac{i}{2}\left(\eta \sigma^{\mu} \overline{\vartheta} - \vartheta \sigma^{\mu} \overline{\eta}\right) \end{aligned} \tag{29}$$

From eq. (29) we read off the infinitesimal action from

$$\begin{aligned} &[\delta_U(\eta, \overline{\eta})] F(\vartheta, \overline{\vartheta}, x) \\ &\sim F_U(\vartheta, \overline{\vartheta}, x) - F(\vartheta, \overline{\vartheta}, x) \\ &= \eta^{\alpha} q_{\alpha} F + \overline{\eta}^{\dot{\alpha}} \overline{q}_{\dot{\alpha}} F \\ q_{\alpha} &= (\partial_{\vartheta})_{\alpha} + \frac{i}{2} \overline{\vartheta}^{\dot{\beta}} (\partial_x)_{\alpha\dot{\beta}} \\ \overline{q}_{\dot{\beta}} &= (\partial_{\overline{\vartheta}})_{\dot{\beta}} + \frac{i}{2} \vartheta^{\alpha} (\partial_x)_{\alpha\dot{\beta}} \\ (\partial_x)_{\alpha\dot{\beta}} &= (\partial_x)^{\mu} \sigma_{\mu\,\alpha\dot{\beta}} \\ (\partial_{\vartheta})_{\alpha} &= \partial / \partial \vartheta^{\alpha} \; ; \; (\partial_{\overline{\vartheta}})_{\dot{\beta}} = \partial / \partial \overline{\vartheta}^{\dot{\beta}} \end{aligned} \tag{30}$$

The pair $q_\alpha\,,\,\overline{q}_{\dot\beta}$ forms by construction a representation of the algebra generated by $Q_\alpha\,,\,Q^*_{\dot\beta}$ (25) where the derivatives $i\,(\partial_x)_\mu \leftrightarrow P_\mu$ play the same role relative to the energy-momentum operator.

$$\begin{aligned} \left\{ q_\alpha\,,\,\overline{q}_{\dot\beta} \right\} &= i\,(\partial_x)_\mu\,\sigma^\mu_{\alpha\dot\beta} \\ \{ q_\alpha\,,\,q_\beta \} &= 0\;;\;\left\{ \overline{q}_{\dot\alpha}\,,\,\overline{q}_{\dot\beta} \right\} = 0 \end{aligned} \tag{31}$$

It may be useful at this point to complete the susy algebra eqs. (25) to the full super-Poincare algebra

$$\begin{aligned} &\left\{ Q_\alpha\,,\,Q^*_{\dot\beta} \right\} = P^\mu\,\sigma_{\mu\,\alpha\dot\beta} \\ &\{ Q_\alpha\,,\,Q_\beta \} = 0\;;\;\left\{ Q^*_{\dot\alpha}\,,\,Q^*_{\dot\beta} \right\} = 0 \\ &\left[P^\mu\,,\,\begin{array}{c} Q_\alpha \\ Q^*_{\dot\beta} \end{array} \right] = 0 \\ &U^{-1}(\mathcal{A})\,Q_\alpha\,U(\mathcal{A}) = \mathcal{A}_{\alpha\beta}\,Q_\beta \\ &U^{-1}(\mathcal{A})\,Q^*_{\dot\alpha}\,U(\mathcal{A}) = \overline{\mathcal{A}}_{\dot\alpha\dot\beta}\,Q^*_{\dot\beta} \end{aligned} \tag{32}$$

In eq. (32) only the extension to fermionic charges of the super-Poincare algebra is displayed.

4c) Base susy differential representation

The base differentials representing (N=1) susy are $\left(q_\alpha , \overline{q}_{\dot\beta} \right)$ **, defined in eqs. (30) and (31) .**

Since the product $U(\eta) F(\vartheta, \overline{\vartheta}, x)$ **, defined in eq. (29) can equally well be interpreted as right-multiplication, commuting with the left one there exist associated differentials** $\left(D_\alpha , \overline{D}_{\dot\beta} \right)$ **, anticommuting with the pair** $\left(q_\alpha , \overline{q}_{\dot\beta} \right)$ **.**

$$\begin{aligned} q_\alpha &= (\partial_\vartheta)_\alpha + \tfrac{i}{2} \overline{\vartheta}^{\dot\beta} (\partial_x)_{\alpha\dot\beta} \\ \overline{q}_{\dot\beta} &= (\partial_{\overline{\vartheta}})_{\dot\beta} + \tfrac{i}{2} \vartheta^\alpha (\partial_x)_{\alpha\dot\beta} \\ \rightarrow \ D_\alpha &= (\partial_\vartheta)_\alpha - \tfrac{i}{2} \overline{\vartheta}^{\dot\beta} (\partial_x)_{\alpha\dot\beta} \\ \overline{D}_{\dot\beta} &= -(\partial_{\overline{\vartheta}})_{\dot\beta} + \tfrac{i}{2} \vartheta^\alpha (\partial_x)_{\alpha\dot\beta} \end{aligned} \tag{33}$$

with the anticommutation relations

$$\begin{aligned} &\left\{ q_\alpha , \overline{q}_{\dot\beta} \right\} = i \, \partial_{\alpha\dot\beta} = \left\{ D_\alpha , \overline{D}_{\dot\beta} \right\} \\ &\left\{ q_\alpha , \begin{array}{c} q_\beta \\ D_\beta \\ \overline{D}_{\dot\beta} \end{array} \right\} = 0 = \left\{ \overline{q}_{\dot\alpha} , \begin{array}{c} \overline{q}_{\dot\beta} \\ D_\beta \\ \overline{D}_{\dot\beta} \end{array} \right\} \end{aligned} \tag{34}$$

and $\left(q_\alpha , \overline{q}_{\dot\beta} \right) \leftrightarrow \left(D_\alpha , \overline{D}_{\dot\beta} \right)$

While $q_A = \left(q_\alpha \, , \, \overline{q}_{\dot\beta} \right) \; ; \; A = 1 \cdots 4$ **define infinitesimal susy transformations,** $D_A = \left(D_\alpha \, , \, \overline{D}_{\dot\beta} \right) \; ; \; A = 1 \cdots 4$ **can be used to set constraints on the substrate of fields on which** q_A **act.**

4d) Explicit construction of chiral (scalar) superfield

In section 3a) (eq. 10) a chiral superfield $\Phi \left(x \, , \, \vartheta_A \right)$ **is introduced, which shall be constructed here explicitely .**

To this end we recall the base quantities $\vartheta_A = \left(\vartheta_\alpha \, , \, \overline{\vartheta}_{\dot\beta} \right)$ **and polynomial invariants**

$$\begin{aligned} &\textbf{base :}\ \theta_\alpha \, , \, \overline{\theta}_{\dot\beta} \ \rightarrow \\ &\theta^\alpha = \varepsilon^{\prime\,\alpha\beta} \, \vartheta_\beta \, , \, \overline{\vartheta}^{\dot\beta} = \varepsilon^{\prime\,\dot\beta\dot\gamma} \, \overline{\vartheta}_{\dot\gamma} \end{aligned} \tag{35}$$

the Grassmann derivatives and raising (lowering) of indices

Derivatives with respect to Grassmann variables shall be denoted by shorthand

$$\begin{aligned} \partial_\alpha = \partial \, / \, \vartheta^\alpha \, , \, \partial^\alpha = \partial \, / \, \vartheta_\alpha \\ \overline{\partial}_{\dot\beta} = \partial \, / \, \overline{\vartheta}^{\dot\beta} \, , \, \overline{\partial}^{\dot\beta} = \partial \, / \, \overline{\vartheta}_{\dot\beta} \end{aligned} \tag{36}$$

Applying the chain rule we find

$$
\begin{aligned}
\partial^{\alpha} &= \Big((\partial / \vartheta_{\alpha}) \, \vartheta^{\beta} \Big) \, \partial_{\beta} \\
&= \Big((\partial / \vartheta_{\alpha}) \, \varepsilon'^{\beta\gamma} \, \vartheta_{\gamma} \Big) \, \partial_{\beta} \\
&= - \varepsilon'^{\alpha\beta} \, \partial_{\beta} \; \rightarrow \qquad (37) \\
\partial^{\alpha} &= - \varepsilon'^{\alpha\beta} \, \partial_{\beta} \; , \; \partial_{\alpha} = - \varepsilon_{\alpha\beta} \, \partial^{\beta} \\
\overline{\partial}^{\dot{\alpha}} &= - \varepsilon'^{\dot{\alpha}\dot{\beta}} \, \partial_{\dot{\beta}} \; , \; \overline{\partial}_{\dot{\alpha}} = - \varepsilon_{\dot{\alpha}\dot{\beta}} \, \overline{\partial}^{\dot{\beta}}
\end{aligned}
$$

[a]

It follows that the raising and lowering of derivative operator components $\left(q_{\alpha} , \overline{q}_{\dot{\beta}} \right)$ and $\left(D_{\alpha} , \overline{D}_{\dot{\beta}} \right)$ is to be performed with the characteristic – sign relative to the base components $\left(\vartheta_{\alpha} , \overline{\vartheta}_{\dot{\beta}} \right)$.

For clarity I list all components for the so defined base pair $\left(D_{\alpha} , \overline{D}_{\dot{\beta}} \right)$

[a] Verify that the – sign in eq. (37) opposite to the base convention of lowering and raising spinor components is correct.

$$
\begin{aligned}
D_{\alpha} &= \partial_{\alpha} - \tfrac{i}{2}\,\overline{\vartheta}^{\,\dot\beta}\,\sigma^{\mu}_{\alpha\dot\beta}\,\partial_{\mu} \\
\overline{D}_{\dot\beta} &= -\,\overline{\partial}_{\dot\beta} + \tfrac{i}{2}\,\vartheta^{\alpha}\,\sigma^{\mu}_{\alpha\dot\beta} \\
D^{\alpha} &= \partial^{\alpha} - \tfrac{i}{2}\,\overline{\vartheta}_{\dot\beta}\,\sigma^{\mu\,\dot\beta\,\alpha}\,\partial_{\mu} \\
\overline{D}^{\,\dot\beta} &= -\,\overline{\partial}^{\,\dot\beta} + \tfrac{i}{2}\,\vartheta_{\alpha}\,\sigma^{\mu\,\dot\beta\,\alpha}\,\partial_{\mu} \\
D^{\alpha} &= -\,\varepsilon^{\prime\,\alpha\beta}\,D_{\beta}\ ,\quad \overline{D}^{\,\dot\beta} = -\,\varepsilon^{\prime\,\dot\beta\dot\alpha}\,\overline{D}_{\dot\alpha} \\
\left\{ D_{\alpha}\,,\,\overline{D}_{\dot\beta} \right\} &= i\,\partial_{\alpha\dot\beta}\ ,\quad \left\{ D^{\alpha}\,,\,\overline{D}^{\,\dot\beta} \right\} = i\,\partial^{\dot\beta\alpha} \\
\partial^{\dot\beta\alpha} &= \varepsilon^{\prime\,\dot\beta\dot\delta}\,\varepsilon^{\prime\,\alpha\gamma}\,\partial_{\gamma\dot\delta}
\end{aligned}
\tag{38}
$$

[a]

Grassmannian Lorentz invariants

Quadratic invariants are formed from $\left(\vartheta_{\alpha}\,,\,\overline{\vartheta}_{\dot\beta} \right)$

[a] Verify the relations in eq. (38) and display the quantities $(\partial_{\alpha\dot\beta}\,,\,\partial^{\dot\beta\alpha})$ by Pauli matrices and the contravariant four vector
$\nabla^{\mu} = (\nabla^{0}\,,\,\vec{\nabla}\,) = (\,\partial_{t}\,,\,-\,\vec{\partial}_{\vec{x}}\,)$.

$$\vartheta_1 = \vartheta^2 , \; \vartheta_2 = - \vartheta^1$$

$$\theta^2 = \tfrac{1}{2} \vartheta^\alpha \vartheta_\alpha = \vartheta_1 \vartheta_2 \rightarrow$$

$$\overline{\theta}^2 = \tfrac{1}{2} \overline{\vartheta}_{\dot\beta} \overline{\vartheta}^{\dot\beta} = \overline{\vartheta}_{\dot 2} \overline{\vartheta}_{\dot 1}$$

$$\text{and } (\partial)^2 = \tfrac{1}{2} \partial^\alpha \partial_\alpha , \; \left(\overline{\partial}\right)^2 = \tfrac{1}{2} \overline{\partial}_{\dot\beta} \overline{\partial}^{\dot\beta} \tag{39}$$

$$(\partial)^2 = (\partial / \partial \vartheta_2)(\partial / \partial \vartheta_1)$$

$$\left(\overline{\partial}\right)^2 = (\partial / \partial \overline{\vartheta}_{\dot 1})(\partial / \partial \overline{\vartheta}_{\dot 2}) \rightarrow$$

$$(\partial)^2 \theta^2 = 1 , \; \left(\overline{\partial}\right)^2 \overline{\theta}^2 = 1$$

From the relations derived in this section (4a-4d eq. (39)) we construct the right-chiral superfield $\Phi (x, \vartheta, \overline{\vartheta})$

$$\textbf{constraint :} \; \overline{D}_{\dot\alpha} \Phi = 0 \rightarrow$$

$$\Phi = \begin{cases} \theta^2 H (x^-) \\ + \vartheta^\alpha \eta_\alpha (x^-) \\ + \varphi (x^-) \end{cases}$$

$$(x^-)^\mu = x^\mu - \tfrac{i}{2} \vartheta^\alpha \sigma^\mu_{\alpha\dot\beta} \overline{\vartheta}^{\dot\beta} ; \; \overline{D}_{\dot\alpha} x^- = 0 \tag{40}$$

From eq. (40) the forms given in eqs. (12-14) follows.

The highest component $|_{\theta^2 \overline{\theta}^2}$ of a general superfield can be projected using the Grassman integration rules and differentials

$$
\begin{aligned}
& d^4 \vartheta = d \overline{\vartheta}_{\dot{1}} \, d \overline{\vartheta}_{\dot{2}} \, d \vartheta_2 \, d \vartheta_1 \\
& \int d \vartheta_\alpha \vartheta_\beta = \delta_{\alpha\beta} \; ; \; \int d \overline{\vartheta}_{\dot{\alpha}} \overline{\vartheta}_{\dot{\beta}} = \delta_{\dot{\alpha}\dot{\beta}}
\end{aligned}
\tag{41}
$$

The 'component form' of the relations in eq. (41) is not explicitely covariant, but could be formally rendered such just writing the index of the differentials as upper index (not 'raising' it) .

We note the identities

$$
\vartheta_\alpha \vartheta_\beta = \varepsilon_{\alpha\beta} \theta^2 \; ; \; \overline{\vartheta}_{\dot{\beta}} \overline{\vartheta}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \overline{\theta}^2 \tag{42}
$$

4e) Chiral invariants from chiral (scalar) superfields (superpotentials)

Lets consider an analytic function of a 'dummy' complex variable z , for which then the right-chiral superfield Φ is substituted

$$
\begin{aligned}
& W(z) \sim \sum_{n=0}^{\infty} w_n z^n \; ; \; \rightarrow \\
& W = W(\Phi) \; ; \; \overline{D}_{\dot{\alpha}} \Phi = 0 \rightarrow \overline{D}_{\dot{\alpha}} W = 0
\end{aligned}
\tag{43}
$$

By the substitution in eq. (43) W becomes a (composite) chiral superfield.

The highest chiral component $W|_{\theta^2}$ represents a susy invariant (modulo a total divergence)

$$W|_{\theta^2} = \int d^2\vartheta\, W \;\rightarrow\; d^2\vartheta = d\vartheta_2\, d\vartheta_1$$
$$W|_{\theta^2} = W_1(\varphi)\,H - \tfrac{1}{2}\,W_2(\varphi)\,\eta^\alpha\eta_\alpha \tag{44}$$
$$W_n(z) = (\partial_z)^n\, W(z)$$

By construction $W|_{\theta^2}$ does not contain space-time derivatives of the components of Φ .

The elementary Wess Zumino model

Considering – only in this subsection – Φ to represent an elementary superfield the most general (perturbatively) renormalizable Lagrangean density takes the form

$$\begin{aligned}
\mathcal{L} &= \overline{\Phi}\,\Phi\Big|_{\theta^2\bar\theta^2} + (W|_{\theta^2} + h.c.)\\
W &= W^{(3)} = b_1 z + \tfrac{1}{2} b_2 z^2 + \tfrac{1}{3!} b_3 z^3\\
W_1 &= b_1 + b_2 z + \tfrac{1}{2} b_3 z^2 \;;\; b_1 \;\rightarrow\; 0\\
W_2 &= b_2 + b_3 z
\end{aligned} \tag{45}$$

We decompose $\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{pot}$ into kinetic and potential parts

and obtain

$$\mathcal{L}_{pot} = H^* H + \Big(W_1 H - \tfrac{1}{2} W_2 \eta^\alpha \eta_\alpha + h.c. \Big)$$
$$\mathcal{L}_{pot} = - V \tag{46}$$

The auxiliary field H is at this stage determined by the extremum condition

$$\delta V / \delta H^* = \delta V / \delta H = 0 \rightarrow$$
$$H = - W_1^* \rightarrow \tag{47}$$
$$V = W_1^* W_1 + \Big(\tfrac{1}{2} W_2 \eta^\alpha \eta_\alpha + h.c. \Big)$$

The extremum yields the maximum of V with respect to (H , H^*) **.**

Now we look for the minimum of V with respect to $\varphi = z + \Delta \varphi$ **, i.e. implying eventually a nontrivial vacuum expected value of the scalar field** φ

$$V_{,z} = W_1^* W_2 \; ; \; V_{,\bar{z}} = W_1 W_2^* \tag{48}$$

From the extremum conditions in eq. (48) we find two solutions

$$a) : W_1 (z) = 0 \; ; \; b) : W_2 (z) = 0 \tag{49}$$

which we inspect in turn.

We do not set $b_1 = 0$ here. It follows

$$a) : W_1(z) = 0 = \frac{1}{2} b_3 (z + r_2)^2 + r_1$$

$$b) : W_2(z) = 0 = b_3 (z + r_2)$$

$$r_2 = b_2 / b_3 \, ; \, r_1 = b_1 - \frac{1}{2} b_2^2 / b_3 \tag{50}$$

It follows from eq. (50) that for case b) we have

$$W_2 = 0 \rightarrow r_1 = 0 \text{ for a minimum of V}$$

$$\rightarrow z = -r_2 \rightarrow W_1(\Delta \varphi) = \frac{1}{2} b_3 (\Delta \varphi)^2$$

$$V = \frac{1}{4} |b_3|^2 |\Delta \varphi|^4 \tag{51}$$

Thus case b) and $r_1 = 0$ correspond to a *unique* **minimum of V and the physical masses of $\Delta \varphi$ and η_α are both 0 .**

For $r_1 = \frac{1}{2} b_3 \varrho^2 \neq 0$ and case a) we have two minima and thus define two secondary vaiables $\Delta_\pm \varphi$:

$$W_1(\Delta \varphi) = \frac{1}{2} b_3 (\Delta \varphi - \varrho)(\Delta \varphi + \varrho)$$

$$\Delta_\pm \varphi = \Delta \varphi \mp \varrho \tag{52}$$

Substituting (either/or) $\Delta_\pm$, W_1 becomes

$$W_1 = \frac{1}{2} b_3 \Delta_\pm \varphi \left(\Delta_\pm \varphi \pm 2 \varrho \right)$$

$$\begin{aligned} V &= \left| b_3 \varrho \right|^2 \left| \Delta_\pm \varphi \right|^2 + \frac{1}{4} \left| b_3 \right|^2 \left| \Delta_\pm \varphi \right|^4 \\ &\quad \pm \frac{1}{2} \left| b_3 \right|^2 \left| \Delta_\pm \varphi \right|^2 \left(\varrho \left(\Delta_\pm \varphi \right)^* + h.c. \right) \end{aligned} \tag{53}$$

The derivations (eqs. 50 - 53) were done under the assumption $b_3 \neq 0$.

The limit $b_3 \rightarrow 0$ can readily be performed. It corresponds to a free theory of a complex scalar φ and a Majorana spinor η_α with a common mass.

However the case $b_3 \neq 0$ always allows to shift $\varphi \rightarrow \Delta_{(\pm)} \varphi$ such that the originally present constant $b_1 \rightarrow 0$. It does describe genuine interactions and still leads in the above semiclassical shift- implementation around any of the two minima of V – to equal masses

$$m = \left| b_3 \varrho \right| \tag{54}$$

for each of the two scalar and fermionic one particle states.

Yet this semiclassical shift is to be proven correct because of the existence of an instanton solution (for $\varrho \neq 0$ and also $b_3 \neq 0$) , interpolating between the two minima of V.

4f) The chiral chain $\Phi \to \Phi'$ or $\Phi_n \to \Phi_{n+1}$

Let the chain start ([1]) with a chiral superfield $\Phi \equiv \Phi_0$ as given in eq. (40)

constraint : $\overline{D}_{\dot\alpha} \Phi = 0 \to$

$$\Phi = \begin{cases} \theta^2 H(x^-) \\ + \vartheta^\alpha \eta_\alpha (x^-) \\ + \varphi(x^-) \end{cases}$$

$$(x^-)^\mu = x^\mu - \tfrac{i}{2} \vartheta^\alpha \sigma^\mu_{\alpha\dot\beta} \overline{\vartheta}^{\dot\beta} \; ; \; \overline{D}_{\dot\alpha} x^- = 0 \tag{55}$$

Then, defining $\overline{D}^2 = \frac{1}{2} \overline{D}_{\dot\alpha} \overline{D}^{\dot\alpha}$, the next chiral superfield along the chain is

$$\left\{ \begin{matrix} \Phi_0 \\ \Phi_n \end{matrix} \right\} \to \left\{ \begin{matrix} \Phi_1 \\ \Phi_{n+1} \end{matrix} \right\} = \overline{D}^2 \left\{ \begin{matrix} \overline{\Phi}_0 \\ \overline{\Phi}_n \end{matrix} \right\}$$

$$\textbf{with :} \quad \Phi_1 = \left\{ \begin{matrix} \theta^2 (-\Box \varphi^*) \\ + \vartheta^\alpha \sigma^{\mu\,\dot\delta}_\alpha i \partial_\mu \eta^*_{\dot\delta} \\ + H^* \end{matrix} \right\} (x^-)$$

$$\Phi_{n+2} = -\Box \Phi_n \; ; \; \overline{D}_{\dot\alpha} \Phi_n = 0 \tag{56}$$

For details of susy transformations only the first two members of the chain are relevant.

5) The N = 1 super-Yang-Mills structure

First we discuss the the susy gauge connection fields, with respect to an arbitrary (hermitian) representation $\mathcal{D}$ of the Lie algebra pertaining to a *simple and compact* **gauge group G**

$$\begin{aligned} &\mathcal{D} \; : \; T^{\,F} \textbf{ with : } \left[\, T^{\,A} \, , \, T^{\,B} \, \right] \; = \; i \; f_{\,ABC} \, T^{\,C} \\ &F \, , \, A \, , \, B \, , \, C \; = \; 1 \, , \, \cdots \, , \, dim \, (\, G \,) \end{aligned} \tag{57}$$

In eq. (57) $f_{\,ABC}$ denote the (totally antisymmetric) structure constants of (Lie-) G, the normalization of which is in general arbitrary, but usually corresponds to some implicit convention. [a]
$dim \, (\, G \,)$ **stands for the (real) dimension of G.**

5a) The $\mathcal{D}$ valued gauge connection from a hermitian vectorfield in the Wess-Zumino gauge

We begin with a hermitian $\mathcal{D}$ valued vector field, which is amputated in the Wess-Zumino gauge to the form

$$\mathcal{D} \; : \; V \; = \; V^{\,A} \, T^{\,A} \tag{58}$$

[a] E.g. for $G \; = \; SU2 : \, f_{\,ABC} \; = \; \varepsilon_{\,ABC} \; , \, dim \, (\, SU2 \,) \; = \; 3 \; .$

$V = V(x)$ in eq. (58) takes the form

$$V = \left\{ \begin{array}{ccc} & \vartheta^2 \overline{\vartheta}^2 D & \\ \vartheta^2 \overline{\vartheta}_{\dot\beta} (\lambda^*)^{\dot\beta} & + & \overline{\vartheta}^2 \vartheta^\alpha \lambda_\alpha \\ & \vartheta^\alpha \overline{\vartheta}^{\dot\beta} v_{\alpha\dot\beta} & \end{array} \right\}$$

$$D = D^A T^A \; ; \; \lambda_\alpha = \lambda_\alpha^A T^A$$

$$v_{\alpha\dot\beta} = v_\mu^A T^A \sigma_{\alpha\dot\beta}^\mu \tag{59}$$

In eq. (59) D^A, v_μ^A denote hermitian fields, whereas the four component spinors

$$\psi_\kappa^A = \begin{pmatrix} \lambda_\alpha \\ \lambda^{*\dot\beta} \end{pmatrix}^A \tag{60}$$

form (for $A = 1, \cdots, dim(G)$) dim G Majorana spinors in the chiral representation, satisfying the hermiticity constraint

$$(\psi = C \gamma_0 \psi^*)^A$$

$$C = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix} ; \; \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \widetilde{\sigma}_\mu & 0 \end{pmatrix} \tag{61}$$

The 2×2 matrices σ_μ, $\widetilde{\sigma}_\mu$ are

$$\sigma_\mu = (\sigma_0, \ \vec{\sigma}) \rightarrow (\sigma_\mu)_{\alpha\dot\beta}$$
$$\widetilde{\sigma}_\mu = (\sigma_0, -\vec{\sigma}) \rightarrow (\sigma_\mu)^{\dot\beta\alpha} \qquad (62)$$

The amputated structure of V implies that only V and V^2 are not vanishing. The latter quantity only has a highest component

$$\tfrac{1}{2} V^2 = \vartheta^2 \overline{\vartheta}^2 v^2$$
$$v^2 = v^{\varrho A} v_\varrho^B \tfrac{1}{2} \left\{ T^A, T^B \right\} \qquad (63)$$

gauge connection

We make the Ansatz

$$W_\alpha = e^{-V} D_\alpha e^V \ ; \ D_\alpha = \partial_\alpha - \tfrac{i}{2} \overline{\vartheta}^{\dot\beta} \partial_{\alpha\dot\beta} \qquad (64)$$

In order to verify susy gauge invariance properties, we choose a set of chiral superfields transforming – component by component – according to the $\mathcal{D}$ - representation of the local gauge group G

$$\mathcal{D} : \Phi \rightarrow \Phi^a$$
$$\left(\Phi^\Omega\right)^a (x) = \Omega^a{}_b (x) \Phi^b$$
$$\left(\Omega = \exp \tfrac{1}{i} \omega^A (x) T^A\right)^a{}_b \qquad (65)$$

in short : $\Phi^\Omega = \Omega \Phi$

In order to preserve chirality and susy covariance we promote each component ω^{A} to a chiral superfield

constraint : $\overline{D}_{\dot{\alpha}} \omega = 0 \; , \; D_{\alpha} \overline{\omega} = 0 \; \rightarrow$

$$\omega = \omega^{A} T^{A} = \begin{cases} \theta^{2} H(x^{-}) \\ + \vartheta^{\alpha} \eta_{\alpha}(x^{-}) \\ + \varphi(x^{-}) \end{cases}$$

$$\overline{\omega} = \overline{\omega}^{A} T^{A}$$

$$\overline{\omega}^{A} = \begin{cases} \overline{\theta}^{2} H^{*A}(x^{+}) \\ + \overline{\vartheta}_{\dot{\alpha}} (\eta^{*})^{\dot{\alpha} A} (x^{+}) \\ + \varphi^{*A}(x^{+}) \end{cases}$$

$$(x^{+})^{\mu} = x^{\mu} + \tfrac{i}{2} \vartheta^{\alpha} \sigma^{\mu}_{\alpha\dot{\beta}} \overline{\vartheta}^{\dot{\beta}} \; ; \; D_{\alpha} x^{+} = 0 \tag{66}$$

Since $(\omega_{(0)} = \varphi)^{A}$ **becomes a complex scalar field, the gauge group is hereby complexified, i.e. extended to complex (infinitesimal) angles.**

$$\rightarrow \begin{array}{l} \Omega = \exp \tfrac{1}{i} \omega \;\; ; \;\; \overline{D}_{\dot{\alpha}} \Omega = 0 \\ \overline{\Omega} = \exp i \overline{\omega} \;\; ; \;\; D_{\alpha} \overline{\Omega} = 0 \end{array} \tag{67}$$

e^{V} as gauge compensator field

Now we generalize the expressions for the kinetic hermitian superfield in eq. (45)

$$\mathcal{L}_{kin} = \overline{\Phi}\, e^{V}\, \Phi = \overline{\Phi}_{a} \left(e^{V} \right)^{a}_{\ b} \Phi^{b}$$

$$\mathcal{L}_{kin} = \mathcal{L}_{kin} (\overline{\Phi} , V , \Phi)$$

susy gauge invariance : $\rightarrow$

$$\mathcal{L}_{kin} (\overline{\Phi^{\Omega}} , V^{\Omega} , \Phi^{\Omega}) = \mathcal{L}_{kin} (\overline{\Phi} , V , \Phi)$$

$$\Phi^{\Omega} = \Omega\, \Phi \, , \quad \overline{\Phi^{\Omega}} = \overline{\Phi}\, \overline{\Omega} \tag{68}$$

It follows for $\exp V^{\Omega}$

$$\begin{aligned} \exp V^{\Omega} &= \exp - \overline{\Omega}\, \exp V\, \exp - \Omega \\ \exp - V^{\Omega} &= \exp \Omega\, \exp - V\, \exp \overline{\Omega} \end{aligned} \tag{69}$$

It is easier to implement eq. (69) for $\exp V^{\Omega}$ **than for** V^{Ω} **itself.**

It follows that the gauge connection (eq. 64) transforms covariantly

$$W^{\Omega}_{\alpha} = \exp \Omega\, W_{\alpha}\, \exp - \Omega \tag{70}$$

Similarly we have for $\overline{W}_{\dot{\alpha}} = - e^{V}\, \overline{D}_{\dot{\alpha}}\, e^{-V}$

$$\overline{W}_{\dot\alpha} = - \exp V \, \overline{D}_{\dot\alpha} \exp - V$$

$$\overline{W}^{\Omega}_{\dot\alpha} = \exp - \overline{\Omega} \, \overline{W}_{\dot\alpha} \exp \overline{\Omega} \quad \leftrightarrow \tag{71}$$

$$W^{\Omega}_{\alpha} = \exp \Omega \, W_{\alpha} \exp - \Omega$$

Because Ω , $\overline{\Omega}$ are right- and left-chiral superfields respectively, acting with $\overline{D}_{\dot\gamma}$ on W_{α} and with D_{γ} on $\overline{W}_{\dot\alpha}$ does not change the gauge transformation properties in eq. (71) .

chiral projection of the gauge connection

Despite interesting properties of the non-chiral fields W_{α} , $\overline{W}_{\dot\alpha}$ we proceed here directly to their chiral projections

$$w_{\alpha} = \overline{D}^{2} W_{\alpha} \leftrightarrow \overline{w}_{\dot\alpha} = D^{2} \overline{W}_{\dot\alpha} \equiv \overline{(w_{\alpha})}$$

$$\overline{D}^{2} = \tfrac{1}{2} \overline{D}_{\dot\gamma} \overline{D}^{\dot\gamma} \; ; \; \overline{D}^{\dot\gamma} = - \left(\varepsilon' \right)^{\dot\gamma\dot\delta} \overline{D}_{\dot\delta}$$

$$\textbf{with :} \quad \overline{D}_{\dot\gamma} \, w_{\alpha} = 0 \leftrightarrow D_{\gamma} \, \overline{w}_{\dot\alpha} = 0 \tag{72}$$

Pro memoria lets recall section 4c) and definitions in eq. (33)

$$D_{\alpha} = (\partial_{\vartheta})_{\alpha} - \tfrac{i}{2} \overline{\vartheta}^{\dot\beta} (\partial_{x})_{\alpha\dot\beta}$$

$$\overline{D}_{\dot\beta} = - (\partial_{\overline{\vartheta}})_{\dot\beta} + \tfrac{i}{2} \vartheta^{\alpha} (\partial_{x})_{\alpha\dot\beta} \tag{73}$$

The chiral fields w_{α} (and $\rightarrow$ $\overline{w}_{\dot{\alpha}}$) become

$$
\begin{aligned}
w_{\alpha} &= w_{\alpha}^{A} T^{A} \\
&= \left\{ \begin{array}{l} \vartheta^{2} \sigma^{\mu}_{\alpha\dot{\beta}} \left(i \partial_{\mu} \lambda^{* \dot{\beta}} - \left[v_{\mu} , \lambda^{* \dot{\beta}} \right] \right) \\ + \vartheta_{\alpha} D + \vartheta^{\gamma} \frac{1}{2} \left(\sigma^{\mu\nu} \right)_{\{ \gamma\alpha \}} \mathcal{F}_{\mu\nu} \\ + \lambda_{\alpha} \end{array} \right\} (x^{-}) \\
\mathcal{W}_{\mu} &= i v_{\mu} \; ; \; \mathcal{W}_{\mu} = i v_{\mu}^{A} T^{A} \\
D_{\mu} &= \partial_{\mu} + [\mathcal{W}_{\mu} , \\
i \partial_{\mu} \lambda^{* \dot{\beta}} &- \left[v_{\mu} , \lambda^{* \dot{\beta}} \right] = i D_{\mu} \lambda^{* \dot{\beta}} \\
\mathcal{F}_{\mu\nu} &= \partial_{\mu} \mathcal{W}_{\nu} - \partial_{\nu} \mathcal{W}_{\mu} + [\mathcal{W}_{\mu} , \mathcal{W}_{\nu}]
\end{aligned}
\tag{74}
$$

In eq. (74) we recognize in D_{μ} the restricted (i.e. non-susy) gauge covariant differential, in $\mathcal{W}_{\mu}$ the correspoding gauge connection and in $\mathcal{F}_{\mu\nu}$ the associated field strength tensor. [a]

The (SL2C-) spin one matrices $(\sigma^{\mu\nu})_{\{ \gamma\alpha \}}$ in eq. (74) are described in the next section (5b)) .

[a] Which are the (anti-) chiral chains associated with w_{α} , $\overline{w}_{\dot{\alpha}}$? (compare with section 4f)) .

5b) The (SL2C-) spin one matrices $\left(\sigma^{\mu\nu} \right)_{\{ \gamma\alpha \}}$

It is useful to consider the mixed spinor indices

$$
\begin{gathered}
\left(\sigma^{\mu\nu} \right)_{\gamma}{}^{\delta} = \left(\varepsilon' \right)^{\delta\alpha} \left(\sigma^{\mu\nu} \right)_{\{ \gamma\alpha \}} \\
\left(\sigma^{\mu} \right)_{\gamma\dot{\beta}} \left(\sigma^{\nu} \right)^{\dot{\beta}\delta} = \eta^{\mu\nu} \delta_{\gamma}{}^{\delta} + \left(\sigma^{\mu\nu} \right)_{\gamma}{}^{\delta} \\
\eta^{\mu\nu} = diag \left(1 , -1 , -1 , -1 \right) \\
\left(\sigma^{\mu\nu} = \left\{ \begin{array}{ll} \sigma_{r} & \textbf{for }^{\mu} = 0 , {}^{\nu} = r \\ \frac{1}{i} \varepsilon_{str} \sigma_{r} & \textbf{for }^{\mu} = s , {}^{\nu} = t \end{array} \right\} \right)_{\gamma}^{\delta}
\end{gathered} \tag{75}
$$

Lets consider the dual to $\sigma^{\mu\nu}$, dropping the mixed spinor indices for simplicity

$$
\begin{gathered}
\widetilde{\sigma}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\varrho\tau} \sigma^{\varrho\tau} \\
\widetilde{\sigma}^{\mu\nu} = \left\{ \begin{array}{ll} i \sigma_{r} & \textbf{for }^{\mu} = 0 , {}^{\nu} = r \\ \varepsilon_{str} \sigma_{r} & \textbf{for }^{\mu} = s , {}^{\nu} = t \end{array} \right\} \\
\rightarrow \widetilde{\sigma}^{\mu\nu} = i \sigma^{\mu\nu}
\end{gathered} \tag{76}
$$

$\left(\sigma , \widetilde{\sigma} \right) \rightarrow \left(\sigma_{R} , \widetilde{\sigma}_{R} \right)$ **defined in eq. (75) bear a (here suppressed) right chiral suffix (R) .**

The corresponding left chiral matrices are

$$\left(\sigma_L^{\mu\nu} \right)^{\dot\gamma}{}_{\dot\delta} = \left(\varepsilon' \right)^{\dot\gamma\dot\alpha} \left(\sigma_L^{\mu\nu} \right)_{\{\dot\alpha\dot\delta\}}$$

$$\left(\sigma^{\mu} \right)^{\dot\gamma\alpha} \left(\sigma^{\nu} \right)_{\alpha\dot\delta} = \eta^{\mu\nu} \delta^{\dot\gamma}{}_{\dot\delta} + \left(\sigma_L^{\mu\nu} \right)^{\dot\gamma}{}_{\dot\delta}$$

$$\left(\sigma_L^{\mu\nu} = \left\{ \begin{array}{ll} -\sigma_r & \textbf{for } ^{\mu} = 0 \, , \, ^{\nu} = r \\ \frac{1}{i} \varepsilon_{str} \sigma_r & \textbf{for } ^{\mu} = s \, , \, ^{\nu} = t \end{array} \right\} \right)^{\dot\gamma}{}_{\dot\delta} \tag{77}$$

Considering the left chiral dual, analogous to the right chiral one in eq. (76) we have, dropping spinor indices

$$\begin{aligned} \widetilde{\sigma}_{\mu\nu\, L} &= \tfrac{1}{2} \varepsilon_{\mu\nu\varrho\tau} \sigma_L^{\varrho\tau} \\ \widetilde{\sigma}_L^{\mu\nu} &= \left\{ \begin{array}{ll} i \sigma_r & \textbf{for } ^{\mu} = 0 \, , \, ^{\nu} = r \\ -\varepsilon_{str} \sigma_r & \textbf{for } ^{\mu} = s \, , \, ^{\nu} = t \end{array} \right\} \\ \rightarrow \quad \widetilde{\sigma}^{\mu\nu\, L} &= -i \sigma^{\mu\nu\, L} \end{aligned} \tag{78}$$

Projecting a photon $F_{\mu\nu}$ onto $\left(\sigma^{\mu\nu} \right)_{R\,(L)}$ we have [a]

$$\tfrac{1}{2} F_{\mu\nu} \left(\sigma^{\mu\nu} \right)_{R\,(L)} = \left\{ \begin{array}{ll} \left(\frac{1}{i} \vec{B} - \vec{E} \right) \vec{\sigma} & \textbf{for R} \\ \left(\frac{1}{i} \vec{B} + \vec{E} \right) \vec{\sigma} & \textbf{for L} \end{array} \right. \tag{79}$$

---

[a] Show that $F \, . \, \sigma_{R\,(L)}$ actually projects onto right (R) - and left (L) - circular photon states respectively .

5c) The chiral field strengths multiplets w_α, $\overline{w}_{\dot\alpha}$ and the chiral Lagrangean multiplet

$$\mathcal{L} = \mathcal{N}\, tr\, w^{\alpha} w_{\alpha}\,,\ \mathcal{N}^{-1} = 4\, C(\mathcal{D})\, g^{2}$$

We repeat the form of w_α given in eq. (74)

$$\begin{aligned}
w_\alpha &= w_\alpha^A T^A \\
&= \left\{ \begin{aligned} &\vartheta^2 \sigma^{\mu}_{\alpha\dot\beta}\, i D_\mu \lambda^{*\dot\beta} \\ &+ \vartheta_\alpha D + \vartheta^\gamma \tfrac{1}{2} (\sigma^{\mu\nu})_{\{\gamma\alpha\}} \mathcal{F}_{\mu\nu} \\ &+ \lambda_\alpha \end{aligned} \right\} (x^-) \\
\mathcal{W}_\mu &= i\, v_\mu\,;\ \mathcal{W}_\mu = i\, v_\mu^A T^A \\
D_\mu &= \partial_\mu + [\,\mathcal{W}_\mu\,, \\
i\,\partial_\mu \lambda^{*\dot\beta} &- \left[v_\mu\,, \lambda^{*\dot\beta} \right] = i\, D_\mu \lambda^{*\dot\beta} \\
\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{W}_\nu - \partial_\nu \mathcal{W}_\mu + [\,\mathcal{W}_\mu\,, \mathcal{W}_\nu\,]
\end{aligned} \tag{80}$$

the adjoint representation

Let the structure constants be defined through the (general irreducible) representation formed by the matrices T^A

$$\begin{aligned} &\left[T^A , T^B \right] = i f_{ABC} T^C \rightarrow \\ &\left(\mathcal{F}^A \right)_{BC} = i f_{BAC} : \\ &\left[\mathcal{F}^A , \mathcal{F}^B \right] = i f_{ABC} \mathcal{F}^C : \mathcal{D} = \mathcal{F} \end{aligned} \tag{81}$$

The matrices $\mathcal{F}^A$ form the adjoint representation of The Lie algebra of G.

The last relation in eq. (81) is equivalent to the Jacobi identity for any triple commutator, e.g.

$$\begin{aligned} &\left[T^A , \left[T^B , T^C \right] \right] \equiv \mathcal{T}^{ABC} \rightarrow \\ &\mathcal{T}^{ABC} + \mathcal{T}^{CAB} + \mathcal{T}^{BCA} = 0 \end{aligned} \tag{82}$$

For any irreducible representation $\mathcal{D}$ it follows that in an appropriate basis for the matrices T^A , $\mathcal{D}$ we have

$$tr\, T^A T^B = C(\mathcal{D})\, \delta_{AB} \tag{83}$$

The normalization constant $C(\mathcal{D})$ is related to the second Casimir invariant pertinent to the representation $\mathcal{D}$

$$\sum_A \left(T^A T^A \right)_{bc} = C_2 (\mathcal{D}) \delta_{bc}$$
$$b , c = 1 , \cdots , dim \mathcal{D} \rightarrow$$
$$C (\mathcal{D}) = C_2 (\mathcal{D}) dim \mathcal{D} / dim G \rightarrow$$
$$C (\mathcal{F}) = C_2 (\mathcal{F}) \tag{84}$$

The absolute normalization of the structure constants f_{ABC} is subject to *convention* **which is illustrated in the section 5d) below, while ratios**

$$C_2 (\mathcal{D}) / C_2 (\mathcal{F}) \tag{85}$$

are structural invariants.

5d) Some numbers for the simple compact Lie groups

P. Minkowski, 3. March, 1994

Below I list the following characteristic numbers for the simple compact Lie groups

$$\textbf{group : G} \quad , \quad \textbf{dimension : } d \quad , \quad \textbf{rank : } r$$

$$\textbf{dimension / rank : } \delta \; = \; d \, / \, r \; , \; \textbf{value of Casimir operator : } C_{\,2}$$

$$\textbf{characteristic number : } \nu \; = \delta \; - \; C_{\,2}$$

The simple compact Lie groups are denoted according to mathematical conventions

$$\begin{array}{c} \textbf{classical groups} \\ \mathbf{A}_{\,n} \; = \; \mathbf{SU\ n{+}1} \quad , \quad \mathbf{B}_{\,n} \; = \; \mathbf{SO\ 2n{+}1} \quad , \quad n \; \geq \; 3 \\ \mathbf{C}_{\,n} \; = \; \mathbf{Sp\ n} \quad , \quad n \; \geq \; 2 \quad ; \quad \mathbf{D}_{\,n} \; = \; \mathbf{SO\ 2n} \quad , \quad n \; \geq \; 4 \\ \textbf{exceptional groups} \\ \mathbf{G}_{\,2} \quad , \quad \mathbf{F}_{\,4} \quad , \quad \mathbf{E}_{\,6} \quad , \quad \mathbf{E}_{\,7} \quad , \quad \mathbf{E}_{\,8} \qquad \longrightarrow \end{array} \tag{86}$$

$\mathbf{G}$	d	r	δ	C_2	ν
$\mathbf{A}_n$	$n(n+2)$	n	$n+2$	$n+1$	1
$\mathbf{B}_n$	$n(2n+1)$	n	$2n+1$	$2n-1$	2
$\mathbf{C}_n$	$n(2n+1)$	n	$2n+1$	$2n+2$	-1
$\mathbf{D}_n$	$n(2n-1)$	n	$2n-1$	$2n-2$	1
$\mathbf{G}_2$	14	2	7	4	3
$\mathbf{F}_4$	52	4	13	9	4
$\mathbf{E}_6$	78	6	13	12	1
$\mathbf{E}_7$	133	7	19	18	1
$\mathbf{E}_8$	248	8	31	30	1

(87)

We give the entriesd in eq. (87) for the ranks of the exceptional groups : 2 , 4 , 6 , 7 , 8 :

$\mathbf{G}$	d	r	δ	C_2	ν
$\mathbf{A}_2$	8	2	4	3	1
$\mathbf{C}_2$	10	2	5	6	-1
$\mathbf{G}_2$	14	2	7	4	3

(88)

$\mathbf{G}$	d	r	δ	C_2	ν
$\mathbf{A}_4$	24	4	6	5	1
$\mathbf{B}_4$	36	4	9	7	2
$\mathbf{C}_4$	36	4	9	10	-1
$\mathbf{D}_4$	28	4	7	6	1
$\mathbf{F}_4$	52	4	13	9	4

(89)

$\mathbf{G}$	d	r	δ	C_2	ν
$\mathbf{A}_6$	48	6	8	7	1
$\mathbf{B}_6$	78	6	13	11	2
$\mathbf{C}_6$	78	6	13	14	-1
$\mathbf{D}_6$	66	6	11	10	1
$\mathbf{E}_6$	78	6	13	12	1

(90)

$\mathbf{G}$	d	r	δ	C_2	ν
$\mathbf{A}_7$	63	7	9	8	1
$\mathbf{B}_7$	105	7	15	13	2
$\mathbf{C}_7$	105	7	15	16	-1
$\mathbf{D}_7$	91	7	13	12	1
$\mathbf{E}_7$	133	7	19	18	1

(91)

$\mathbf{G}$	d	r	δ	C_2	ν
$\mathbf{A}_8$	80	8	10	9	1
$\mathbf{B}_8$	136	8	17	15	2
$\mathbf{C}_8$	136	8	17	18	-1
$\mathbf{D}_8$	120	8	15	14	1
$\mathbf{E}_8$	248	8	31	30	1

(92)

We note, that Élie Cartan [2] constructed the large exceptional groups first from his alphabetical ordering A , B , C , D , E , F , G .

5c) continued, w_{α} , $\overline{w}_{\dot{\alpha}}$ and the chiral Lagrangean multiplet

$\mathcal{L} = \mathcal{N} \, tr \, w^{\alpha} w_{\alpha}$, $\mathcal{N}^{-1} = 4 \, C (\mathcal{D}) \, g^{2}$

We consider the ϑ^{2} component of the chiral bilinear

$$\begin{aligned}
& (\mathcal{N}^{-1} \mathcal{L} = tr \, w^{\alpha} w_{\alpha})_{\vartheta^{2}} = \\
& 2 \, C (\mathcal{D}) \, \vartheta^{2} \times \\
& \times \left[\begin{array}{c}
\lambda^{* \, B}_{\dot{\beta}} \left\{ \begin{array}{c} i \overleftarrow{\partial}_{\mu} \, \delta_{BA} \\ + \, i \, \mathcal{W}^{C}_{\mu} \left(\mathcal{F}^{C} \right)_{BA} \end{array} \right\} \sigma^{\mu \, \dot{\beta} \alpha} \lambda^{A}_{\alpha} \\
+ \, D^{A} D^{A} \\
- \left(\vec{B}^{A} - i \, \vec{E}^{A} \right) \left(\vec{B}^{A} - i \, \vec{E}^{A} \right)
\end{array} \right]
\end{aligned} \tag{93}$$

defining the hermition field strengths $F_{\mu\nu} \rightarrow F^{A}_{\mu\nu}$ and associated electric $E^{A \, k} \rightarrow \vec{E}^{A}$ and magnetic $B^{A \, k} \rightarrow \vec{B}^{A}$ ($k = 1, 2, 3$) field components

$$\begin{aligned}
& i \, \mathcal{F}_{\mu\nu} = \partial_{\nu} v_{\mu} - \partial_{\mu} v_{\nu} + i \, [\, v_{\nu} \, , \, v_{\mu} \,] = F_{\mu\nu} \\
& F_{\mu\nu} = F^{A}_{\mu\nu} \, T^{A} \\
& F^{A}_{\mu\nu} = \partial_{\nu} v^{A}_{\mu} - \partial_{\mu} v^{A}_{\nu} - f_{ABC} \, v^{B}_{\nu} \, v^{C}_{\mu} \\
& E^{A \, k} = F^{A \, 0k} \; , \; B^{A \, k} = \tfrac{1}{2} \, \varepsilon_{kmn} \, F^{A \, mn}
\end{aligned} \tag{94}$$

The symbol $X \overleftarrow{\partial}_{\mu}$ in eq. (93) denote the derivative acting with a minus sign to the left

$$X \overleftarrow{\partial}_{\mu} = - \partial_{\mu} X \tag{95}$$

We rescale the fermion fields $\lambda_{\alpha}^{A} = \sqrt{2}\, \Lambda_{\alpha}^{A}$ and cast eq. (93) to the form

$$\begin{aligned}
&(\mathcal{N}^{-1} \mathcal{L} = tr\, w^{\alpha} w_{\alpha})_{\vartheta^{2}} = \\
&4\, C (\mathcal{D})\, \vartheta^{2} \times \\
&\times \left[\begin{array}{c}
\Lambda_{\dot{\beta}}^{*\,B} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_{\mu} \delta_{BA} \\ + i\, \mathcal{W}_{\mu}^{C} \left(\mathcal{F}^{C} \right)_{BA} \end{array} \right\} \sigma^{\mu\, \dot{\beta}\alpha} \Lambda_{\alpha}^{A} \\
- \frac{i}{2} \partial_{\mu} \left(\Lambda_{\dot{\beta}}^{*\,A} \sigma^{\mu\, \dot{\beta}\alpha} \Lambda_{\alpha}^{A} \right) \\
+ \frac{1}{2} D^{A} D^{A} \\
+ \frac{1}{2} \left(\vec{E}^{A} \vec{E}^{A} - \vec{B}^{A} \vec{B}^{A} \right) \\
+ i \left(\vec{E}^{A} \vec{B}^{A} \right)
\end{array} \right]
\end{aligned} \tag{96}$$

The red entries to $\mathcal{L}$ in eq. (96) are antihermitian , whereas the remaining ones are hermitian.

We rewrite the Lagrangean multiplet component in eq. (96) in terms of the field strength tensor and the overall chiral fermion current $j_{\Lambda}^{\mu} = \Lambda_{\dot{\beta}}^{*\,A} \sigma^{\mu\, \dot{\beta}\alpha} \Lambda_{\alpha}^{A}$

$$\Phi = (4 \, C (\mathcal{D}))^{-1} \, tr \, w^{\alpha} \, w_{\alpha} =$$

$$\left\{ \begin{array}{l} \vartheta^{2} \left[\begin{array}{c} \overset{*}{\Lambda}{}_{\dot\beta}^{\ B} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_{\mu} \, \delta_{BA} \\ - v_{\mu}^{\ C} \left(\mathcal{F}^{C} \right)_{BA} \end{array} \right\} \sigma^{\mu \, \dot\beta\alpha} \, \Lambda_{\alpha}^{\ A} \\ - \frac{i}{2} \, \partial_{\mu} \, j_{\Lambda}^{\ \mu} \\ + \frac{1}{2} \, D^{A} \, D^{A} \\ - \frac{1}{4} \, F^{\mu\nu \ A} \left(F_{\mu\nu}^{\ A} - i \, \widetilde{F}_{\mu\nu}^{\ A} \right) \end{array} \right] \\ + \vartheta^{\alpha} \frac{1}{\sqrt{2}} \left[D^{A} \, \Lambda_{\alpha}^{\ A} + \mathcal{F}_{\alpha}^{\ A \ \gamma} \Lambda_{\gamma}^{\ A} \right] \\ + \frac{1}{2} \, \Lambda^{A \ \alpha} \, \Lambda_{\alpha}^{\ A} \end{array} \right.$$

$$j_{\Lambda}^{\ \mu} = \overset{*}{\Lambda}{}_{\dot\beta}^{\ A} \, \sigma^{\mu \, \dot\beta\alpha} \, \Lambda_{\alpha}^{\ A} \; ; \; \mathcal{F}_{\alpha}^{\ A \ \gamma} = \frac{1}{2} \, \mathcal{F}_{\mu\nu} \left(\sigma^{\mu\nu} \right)_{\alpha}^{\ \gamma}$$

$$X \overleftrightarrow{\partial}_{\mu} Y = - (\partial_{\mu} X) \, Y + X (\partial_{\mu} Y) \tag{97}$$

The ϑ^{2} component of Φ in eq. (97) restricted to its hermition part and multiplied wit $1 / g^{2}$ serves as starting point Lagrangean density to the N=1 super-Yang-Mills structure.

6) The minimal source extension of N=1 super-Yang-Mills structure [3]

We consider an external chiral multiplet , called the (space-time dependent -) coupling constant multiplet, denoted by $J(y)$

$$\Phi = \left\{ \begin{array}{l} \vartheta^2 H \\ \\ + \vartheta^\alpha \eta_\alpha \\ \\ + \varphi \end{array} \right\} (x^-) \leftrightarrow$$

$$J = \left\{ \begin{array}{l} \vartheta^2 (-m) \\ \\ + \vartheta^\alpha \psi_\alpha \\ \\ + j \end{array} \right\} (x^-) \tag{98}$$

$$j(y) = \frac{1}{2} \left[(1/g^2(y)) - \frac{i}{8\pi^2} \Theta(y) \right]$$

The external sources contained in $J(y)$ are thus the four complex functions

$$J(y) \rightarrow (m, \psi_\alpha, j)(y) \tag{99}$$

The Lagrangean density, denoted L, is defined by the source dependent products

$$L(J) = \int d^2\vartheta \, (\Phi J) + h.c. \tag{100}$$

The chiral multiplets shall be denoted

$$\mathcal{L} (J) = \Phi J , \overline{\mathcal{L}} (\overline{J}) = \overline{J} \overline{\Phi} \tag{101}$$

The Lagrangean in the absence of sources involves the 'limiting conditions'

$$\lim_{x \to \infty} J (x) = J_{\infty} :$$

$$\lim_{x \to \infty} \begin{cases} m (x) & = m_{\infty} \to 0 \\ \psi_{\alpha} (x) & = \psi_{\alpha \infty} \to 0 \\ g (x) & = g_{\infty} \\ \Theta (x) & = \Theta_{\infty} \end{cases} \tag{102}$$

and the relative sources

$$J_{\Delta} (x) = J (x) - J_{\infty} \to 0 \tag{103}$$

whereby the magenta $\to 0$ limit (eq. 103) is to be performed first and the red $\to 0$ limit (eq. 102) last.

We proceed to calculate the multiplets $\mathcal{L}$, $\overline{\mathcal{L}}$: first the ϑ^{2} components

$$\mathcal{L}(J)|_{\vartheta^2} = H\, j - \eta^{\alpha}\psi_{\alpha} - m\,\varphi$$

$$H = \left[\begin{array}{c} \Lambda^{*\,B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2}\overleftrightarrow{\partial}_{\mu}\,\delta_{BA} \\ -\,v^{C}_{\mu}\left(\mathcal{F}^{C}\right)_{BA} \end{array} \right\} \sigma^{\mu\,\dot\beta\alpha}\,\Lambda^{A}_{\alpha} \\ -\,\frac{i}{2}\,\partial_{\mu}\, j^{\mu}_{\Lambda} \\ +\,\frac{1}{2}\, D^{A} D^{A} \\ -\,\frac{1}{4}\, F^{\mu\nu\,A}\left(F^{A}_{\mu\nu} - i\,\widetilde{F}^{A}_{\mu\nu}\right) \end{array} \right]$$

$$j = \frac{1}{2}\left[\left(1/g^{2} - \frac{i}{8\pi^{2}}\,\Theta\right]\right.$$

$$j^{\mu}_{\Lambda} = \Lambda^{*\,A}_{\dot\beta}\,\sigma^{\mu\,\dot\beta\alpha}\,\Lambda^{A}_{\alpha}$$

$$\eta_{\alpha} = \frac{1}{\sqrt{2}}\left[D^{A}\,\Lambda^{A}_{\alpha} + \mathcal{F}^{A\,\gamma}_{\alpha}\,\Lambda^{A}_{\gamma}\right]$$

$$\varphi = \frac{1}{2}\,\Lambda^{A\,\alpha}\,\Lambda^{A}_{\alpha} \tag{104}$$

From eq. (104) we infer the Lagrangean density (eq. 100)

$$L(J) =$$

$$\Bigg[\frac{1}{g^2} \Lambda^{*\,B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_\mu \, \delta_{BA} \\ - v^C_\mu \left(\mathcal{F}^C \right)_{BA} \end{array} \right\} \sigma^{\mu\,\dot\beta\alpha} \Lambda^A_\alpha$$
$$- \frac{1}{4 g^2} F^{\mu\nu\,A} F^A_{\mu\nu} + \frac{1}{2 g^2} D^A D^A$$
$$+ \frac{1}{8\pi^2} \Theta \left(\frac{1}{4} F^{\mu\nu\,A} \widetilde{F}^A_{\mu\nu} - \frac{1}{2} \partial_\mu j^\mu_\Lambda \right)$$
$$+ \frac{1}{\sqrt{2}} \left[D^A \Lambda^A_\alpha + \mathcal{F}^{A\,\gamma}_\alpha \Lambda^A_\gamma \right] \psi^\alpha + h.c.$$
$$- \frac{1}{2} m \Lambda^{A\,\alpha} \Lambda^A_\alpha - h.c. \Bigg] \tag{105}$$

Upon retaining the fermionic source ψ_α the elimination of the auxiliary fields D^A leaves the latter nontrivial

$$\delta L(J) / \delta D^A =$$
$$= \frac{1}{g^2} D^A + \frac{1}{\sqrt{2}} \left(\Lambda^A_\alpha \psi^\alpha + h.c. \right) \to 0 \tag{106}$$

The net term induced through D^A becomes

$$L_D(J) = - \frac{g^2}{4} \left[\begin{array}{c} \left(\Lambda^A_\alpha \psi^\alpha + \psi^{*\,\dot\beta} \Lambda^{*\,A}_{\dot\beta} \right) \times \\ \times \left(\Lambda^A_\gamma \psi^\gamma + \psi^{*\,\dot\delta} \Lambda^{*\,A}_{\dot\delta} \right) \end{array} \right] \tag{107}$$

We reorder the source fermion fields ψ_α , $\psi^*{}_{\dot\beta}$ in eq. (107) all to the left

$$L_D(J) =$$

$$= -\frac{g^2}{4}\left[\begin{array}{l} (\psi_\alpha \psi^\gamma)\left(\Lambda^{A\alpha}\Lambda^A_\gamma\right) + h.c. \\ +2\left(\psi^{*\dot\delta}\psi^\alpha\right)\left(\Lambda^{*A}_{\dot\delta}\Lambda^A_\alpha\right)\end{array}\right] \tag{108}$$

and defining in an equivalent way to ϑ^2 , $\overline{\vartheta}^2$ in eq. (39) the Lorentz invariants pertaining to the fermionic sources

$$\begin{aligned} \psi^2 &= \tfrac{1}{2}\psi^\alpha\psi_\alpha = \psi_1\psi_2 \\ \overline{\psi}^2 &= \tfrac{1}{2}\psi^*_{\dot\beta}\psi^{*\dot\beta} = \psi^*_{\dot 2}\psi^*_{\dot 1} \end{aligned} \tag{109}$$

we obtain

$$L_D(J) =$$

$$= -\frac{g^2}{4}\left[\begin{array}{l} -\psi^2\Lambda^{A\alpha}\Lambda^A_\alpha + h.c. \\ +2\left(\psi^{*\dot\delta}\psi^\alpha\right)\left(\Lambda^{*A}_{\dot\delta}\Lambda^A_\alpha\right)\end{array}\right] \tag{110}$$

The last term on the right hand side of eqs. (108 and 110) represents a hermitian current-current coupling. We use the identity

$$\sigma^{\mu\,\dot{\delta}\alpha}\,\sigma_{\mu\,\beta\dot{\gamma}} \;=\; 2\,\delta^{\,\dot{\delta}}_{\,\dot{\gamma}}\,\delta^{\,\alpha}_{\,\beta} \tag{111}$$

whereupon $L_D(J)$ takes the form

$$L_D(J) =$$
$$= -\frac{g^2}{4}\left[\,-\psi^2\,\Lambda^{A\,\alpha}\,\Lambda^{A}_{\alpha} + \frac{1}{2}\,j^{\mu}_{\psi}\,j_{\mu\,\Lambda}\,\right] + h.c.$$
$$j^{\mu}_{\psi} \;=\; \psi^{*}_{\dot{\delta}}\,\sigma^{\mu\,\dot{\delta}\alpha}\,\psi_{\alpha}$$
$$j^{\mu}_{\Lambda} \;=\; \Lambda^{*\,A}_{\dot{\delta}}\,\sigma^{\mu\,\dot{\delta}\alpha}\,\Lambda^{A}_{\alpha} \tag{112}$$

We rewrite $L(J)$ in eq. (105) exhibiting sources in red .

$$L(J) = L_{div} + L_1(J)$$

$$L_1(J) =$$

$$\left[\begin{array}{l} \frac{1}{g^2} \Lambda^{*\,B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_\mu \delta_{BA} \\ - v^C_\mu \left(\mathcal{F}^C \right)_{BA} \end{array} \right\} \sigma^{\mu\,\dot\beta\alpha} \Lambda^A_\alpha \\ \qquad + \frac{1}{g^2} \left(-\frac{1}{4} F^{\mu\nu\,A} F^A_{\mu\nu} \right) \\ \qquad + \frac{1}{8\pi^2} \Theta \left(\frac{1}{4} F^{\mu\nu\,A} \widetilde{F}^A_{\mu\nu} \right) \\ \qquad + \left(\frac{1}{16\pi^2} \partial_\mu \Theta - \frac{g^2}{4} j_{\mu\,\psi} \right) j^\mu_\Lambda \\ \qquad - \frac{1}{\sqrt{2}} \left[\psi^\alpha \mathcal{F}^{A\,\gamma}_\alpha \Lambda^A_\gamma \right] + h.c. \\ \qquad - \frac{1}{2} \left(m - \frac{g^2}{4} \psi^2 \right) \Lambda^{A\,\alpha} \Lambda^A_\alpha + h.c. \end{array} \right]$$

$$L_{div} = -\partial_\mu \left(\frac{1}{16\pi^2} \Theta j^\mu_\Lambda \right) \tag{113}$$

6a) The minimal source extension of the D^{A}-eliminated Lagrangean multiplet for N=1 super-Yang-Mills

It is obviously important to reconstruct the chiral Lagrangean multiplet $\mathcal{L}(J)$ as defined in eqs. (98 - 101) , taking into account the elimination of the auxiliary fields D^{A} .

We turn to this task next, repeating the chiral multiplet structures pertaining to dynamical fields Φ and sources J

$$\Phi = \left\{ \begin{array}{l} \vartheta^{2} H \\ + \vartheta^{\alpha} \eta_{\alpha} \\ + \varphi \end{array} \right\} (x^{-}) \leftrightarrow$$

$$J = \left\{ \begin{array}{l} \vartheta^{2} (-m) \\ + \vartheta^{\alpha} \psi_{\alpha} \\ + j \end{array} \right\} (x^{-}) \tag{114}$$

$$j(y) = \frac{1}{2} \left[(1/g^{2}(y)) - \frac{i}{8\pi^{2}} \Theta(y) \right]$$

$$\eta_{\alpha} = \frac{1}{\sqrt{2}} \left[D^{A} \Lambda^{A}_{\alpha} + \mathcal{F}^{A}_{\alpha}{}^{\gamma} \Lambda^{A}_{\gamma} \right]$$

The fermion field η_{α} contains the auxiliary fields (eq. 104) and becomes using eq. (106) expanded below

$$\delta L(J)/\delta D^{A} =$$
$$= \frac{1}{g^{2}} D^{A} + \frac{1}{\sqrt{2}} \left(\Lambda^{A}_{\alpha} \psi^{\alpha} + h.c. \right) \rightarrow 0 \quad (115)$$
$$D^{A} = \frac{g^{2}}{\sqrt{2}} \left(\psi^{\alpha} \Lambda^{A}_{\alpha} + \Lambda^{*A}_{\dot\beta} \psi^{*\dot\beta} \right)$$

Thus η_{α} becomes (eq. 114)

$$\eta_{\alpha} = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \mathcal{F}^{A\ \gamma}_{\alpha} \Lambda^{A}_{\gamma} + \\ \frac{g^{2}}{2} \left(\psi^{\gamma} \Lambda^{A}_{\gamma} + \Lambda^{*A}_{\dot\beta} \psi^{*\dot\beta} \right) \Lambda^{A}_{\alpha} \end{array} \right] \quad (116)$$

We go step by step to avoid sign errors

$$\eta_{\alpha} =$$
$$= \left[\begin{array}{c} \frac{1}{\sqrt{2}} \mathcal{F}^{A\ \gamma}_{\alpha} \Lambda^{A}_{\gamma} \\ -\frac{g^{2}}{2} \left(\psi_{\gamma} \Lambda^{A\gamma} \Lambda^{A}_{\alpha} + \psi^{*\dot\beta} \Lambda^{*A}_{\dot\beta} \Lambda^{A}_{\alpha} \right) \end{array} \right] \quad (117)$$

First we use the identity

$$\Lambda^{A\gamma} \Lambda^{A}_{\alpha} = \frac{1}{2} \delta^{\gamma}_{\alpha} \Lambda^{A\beta} \Lambda^{A}_{\beta}$$

and obtain

$$\eta_{\alpha} = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \mathcal{F}_{\alpha}^{\ A \ \gamma} \Lambda_{\gamma}^{\ A} \\ - \frac{g^{\ 2}}{4} \psi_{\alpha} \Lambda^{A \ \delta} \Lambda_{\delta}^{\ A} \\ - \frac{g^{\ 2}}{4} \psi^{* \ \dot{\gamma}} \Lambda_{\dot{\beta}}^{* \ A} \Lambda_{\delta}^{\ A} \left(2 \, \delta_{\dot{\gamma}}^{\ \dot{\beta}} \delta_{\alpha}^{\ \delta} \right) \end{array} \right] \tag{118}$$

Next we substitute (eq. 111)

$$2 \, \delta_{\dot{\gamma}}^{\ \dot{\beta}} \delta_{\alpha}^{\ \delta} = \sigma^{\mu \ \dot{\beta}\delta} \sigma_{\mu \ \alpha\dot{\gamma}}$$

whereby η_{α} **becomes**

$$\eta_{\alpha} = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \mathcal{F}_{\alpha}^{\ A \ \gamma} \Lambda_{\gamma}^{\ A} \\ - \frac{g^{\ 2}}{4} \psi_{\alpha} \Lambda^{A \ \delta} \Lambda_{\delta}^{\ A} \\ - \frac{g^{\ 2}}{4} \psi_{\mu \ \alpha}^{*} j_{\Lambda}^{\ \mu} \end{array} \right] \tag{119}$$

$$\psi_{\mu \ \alpha}^{*} = \psi^{* \ \dot{\gamma}} \sigma_{\mu \ \alpha\dot{\gamma}}$$

$$j_{\Lambda}^{\ \mu} = \Lambda_{\dot{\beta}}^{* \ A} \sigma^{\mu \ \dot{\beta}\delta} \Lambda_{\delta}^{\ A}$$

Now we turn to the θ^{2} component of Φ (eq. 104) , where D^{A} are to be substituted (eq. 115)

$$H = \left[\begin{array}{c} \Lambda^{*\,B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_{\mu} \, \delta_{BA} \\ - v^{C}_{\mu} \left(\mathcal{F}^{C} \right)_{BA} \end{array} \right\} \sigma^{\mu\,\dot\beta\alpha} \Lambda^{A}_{\alpha} \\ - \frac{i}{2} \, \partial_{\mu} \, j^{\mu}_{\Lambda} \\ + \frac{1}{2} D^{A} D^{A} \\ - \frac{1}{4} F^{\mu\nu\,A} \left(F^{A}_{\mu\nu} - i \widetilde{F}^{A}_{\mu\nu} \right) \end{array} \right]$$

$$D^{A} = \frac{g^{2}}{\sqrt{2}} \left(\psi^{\alpha} \Lambda^{A}_{\alpha} + \Lambda^{*\,A}_{\dot\beta} \psi^{*\,\dot\beta} \right) \tag{120}$$

Thus we proceed to evaluate the quantity $\frac{1}{2} D^{A} D^{A}$

$$\frac{1}{2} D^{A} D^{A} = $$
$$= \frac{g^{4}}{4} \left(\begin{array}{c} \psi^{\alpha} \psi_{\gamma} \Lambda^{A}_{\alpha} \Lambda^{A\,\gamma} + h.c. \\ + \psi^{*\,\dot\beta} \psi^{\alpha} \, 2 \, \Lambda^{*\,A}_{\dot\beta} \Lambda^{A}_{\alpha} \end{array} \right) \tag{121}$$

and using the identity in eq. (111) it follows

$$\frac{1}{2} D^{A} D^{A} = $$
$$= \frac{g^{4}}{4} \left(\begin{array}{c} - \psi^{2} \Lambda^{A\,\alpha} \Lambda_{A\,\alpha} + h.c. \\ + j_{\mu\,\psi} \, j^{\mu}_{\Lambda} \end{array} \right) \tag{122}$$

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6b) The minimal source extension D^{A}-eliminated Φ multiplet (results)

The expressions for $\Phi = (H , \eta_{\alpha} , \varphi)$ in eq. (114) become (eqs. 104 , 119 , 122)

$$H = \left[\begin{array}{c} \Lambda^{*\,B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_{\mu} \delta_{BA} \\ - v^{C}_{\mu} \left(\mathcal{F}^{C} \right)_{BA} \end{array} \right\} \sigma^{\mu\,\dot\beta\alpha} \Lambda^{A}_{\alpha} \\ - \frac{i}{2} \partial_{\mu} j^{\mu}_{\Lambda} \\ - \frac{1}{4} F^{\mu\nu\,A} \left(F^{A}_{\mu\nu} - i \widetilde{F}^{A}_{\mu\nu} \right) \\ + \frac{g^{4}}{4} \left(\begin{array}{c} - \psi^{2} \Lambda^{A\,\alpha} \Lambda_{A\,\alpha} + h.c. \\ + j_{\mu\,\psi} j^{\mu}_{\Lambda} \end{array} \right) \end{array} \right]$$

$$\eta_{\alpha} = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \mathcal{F}^{A\,\gamma}_{\alpha} \Lambda^{A}_{\gamma} \\ - \frac{g^{2}}{4} \psi_{\alpha} \Lambda^{A\,\delta} \Lambda^{A}_{\delta} \\ - \frac{g^{2}}{4} \psi^{*}_{\mu\,\alpha} j^{\mu}_{\Lambda} \end{array} \right]$$

$$\varphi = \frac{1}{2} \Lambda^{A\,\alpha} \Lambda^{A}_{\alpha} \tag{123}$$

The antihermitian parts of H in eq. (123) are marked with a color magenta factor i .

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6c) The minimal source extension D^A-eliminated Lagrangean multiplet $\mathcal{L}(J)$

Having established the D^A-eliminated form of Φ in eq. (123) , we go back to the external source multiplet (eq. 98) and Φ , reproduced below

$$\Phi = \left\{ \begin{array}{l} \vartheta^2 H \\ + \vartheta^\alpha \eta_\alpha \\ + \varphi \end{array} \right\} (x^-) \leftrightarrow$$

$$J = \left\{ \begin{array}{l} \vartheta^2 (-m) \\ + \vartheta^\alpha \psi_\alpha \\ + j \end{array} \right\} (x^-) \tag{124}$$

$$j(y) = \frac{1}{2} \left[(1/g^2(y)) - \frac{i}{8\pi^2} \Theta(y) \right]$$

The next step is to determine the Lagrangean multiplet $\mathcal{L}(J) = \Phi J$

$$\mathcal{L}(J) = \left\{ \begin{array}{l} \vartheta^2 A \\ + \vartheta^\alpha B_\alpha \\ + C \end{array} \right\} (x^-) \tag{125}$$

The components $\mathcal{L}(J) = (A, B_\alpha, C)$ **are**

$$\begin{aligned} A &= j\,H - \psi^\alpha \eta_\alpha - m\,\varphi \\ B_\alpha &= j\,\eta_\alpha + \psi_\alpha \varphi \\ C &= j\,\varphi \end{aligned} \tag{126}$$

$$L(J) = A + A^*$$

$$j(y) = \frac{1}{2}\left[\frac{1}{g^2}(y) - \frac{i}{8\pi^2}\Theta(y)\right]$$

We first give $j\,H$ **(eq. 123)**

$$j\,H = A_1 =$$

$$\left[\begin{array}{c} \left(\frac{1}{2g^2} - \frac{i}{16\pi^2}\Theta\right)\times \\ \times\, \Lambda^{*B}_{\dot\beta} \left\{\begin{array}{c} \frac{i}{2}\overleftrightarrow{\partial}_\mu\,\delta_{BA} \\ - v^C_\mu\left(\mathcal{F}^C\right)_{BA} \end{array}\right\} \sigma^{\mu\,\dot\beta\alpha}\Lambda^A_\alpha \\ - \left(\frac{1}{32\pi^2}\Theta + \frac{i}{4g^2}\right)\partial_\mu\, j^\mu_\Lambda \\ - \left(\frac{1}{8g^2} - \frac{i}{64\pi^2}\Theta\right) F^{\mu\nu\,A}\left(F^A_{\mu\nu} - i\,\widetilde{F}^A_{\mu\nu}\right) \\ + \left(\frac{g^2}{8} - \frac{i}{64\pi^2}g^4\,\Theta\right)\times \\ \times \left(\begin{array}{c} -\psi^2\Lambda^{A\alpha}\Lambda_{A\alpha} + h.c. \\ + j_{\mu\psi}\, j^\mu_\Lambda \end{array}\right) \end{array}\right] \tag{127}$$

Next we determine $\psi^{\alpha} \eta_{\alpha}$ contributing with a negative sign to A repeating eq. (123) below

$$\eta_{\alpha} = \left[\begin{array}{c} \frac{1}{\sqrt{2}} \mathcal{F}_{\alpha}^{A \gamma} \Lambda_{\gamma}^{A} \\ - \frac{g^{2}}{4} \psi_{\alpha} \Lambda^{A \delta} \Lambda_{\delta}^{A} \\ - \frac{g^{2}}{4} \psi^{*}_{\mu \alpha} j_{\Lambda}^{\mu} \end{array} \right] \rightarrow$$

$$\psi^{\alpha} \eta_{\alpha} = - A_{2} = \tag{128}$$

$$= \left[\begin{array}{c} \frac{1}{\sqrt{2}} \psi^{\alpha} \mathcal{F}_{\alpha}^{A \gamma} \Lambda_{\gamma}^{A} \\ - \frac{g^{2}}{2} \psi^{2} \Lambda^{A \delta} \Lambda_{\delta}^{A} \\ + \frac{g^{2}}{4} j_{\mu \psi} j_{\Lambda}^{\mu} \end{array} \right]$$

Finally (for A) we turn to $m \varphi$ which is independent of D^{A}

$$m \varphi = - A_{3} = \frac{1}{2} m \Lambda^{A \alpha} \Lambda_{\alpha}^{A} \tag{129}$$

Hence we obtain $A = \sum_{k}^{3} A_{k}$ (eqs. 127 - 129)

$$
A = \sum_k^3 A_k =
$$

$$
\left[
\begin{array}{c}
\left(\frac{1}{2 g^2} - \frac{i}{16 \pi^2} \Theta \right) \times \\
\times \Lambda^{* B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_\mu \delta_{BA} \\ - v^C_\mu \left(\mathcal{F}^C \right)_{BA} \end{array} \right\} \sigma^{\mu \dot\beta \alpha} \Lambda^A_\alpha \\
- \left(\frac{1}{32 \pi^2} \Theta + \frac{i}{4 g^2} \right) \partial_\mu j^\mu_\Lambda \\
- \left(\frac{1}{8 g^2} - \frac{i}{64 \pi^2} \Theta \right) F^{\mu\nu A} \left(F^A_{\mu\nu} - i \widetilde{F}^A_{\mu\nu} \right) \\
+ \left(\frac{g^2}{8} - \frac{i}{64 \pi^2} g^4 \Theta \right) \times \\
\times \left(\begin{array}{c} - \psi^2 \Lambda^{A \alpha} \Lambda_{A \alpha} + h.c. \\ + j_{\mu \psi} j^\mu_\Lambda \end{array} \right) \\
+ \frac{g^2}{2} \psi^2 \Lambda^{A \alpha} \Lambda^A_\alpha - \frac{g^2}{4} j_{\mu \psi} j^\mu_\Lambda \\
- \frac{1}{\sqrt{2}} \psi^\alpha \mathcal{F}^{A \gamma}_\alpha \Lambda^A_\gamma - \frac{1}{2} m \Lambda^{A \alpha} \Lambda^A_\alpha
\end{array}
\right]
$$

result for A

$$
\tag{130}
$$

We repeat the structure of $\mathcal{L}(J)$ (eq. 125)

$$
\mathcal{L}(J) = \left\{ \begin{array}{l} \vartheta^2 A \\ + \vartheta^\alpha B_\alpha \\ + C \end{array} \right\} (x^-) \tag{131}
$$

Next we turn to the fermionic components B_α of $\mathcal{L}(J)$ (eq. 126)

$$B_\alpha = j\,\eta_\alpha + \psi_\alpha\,\varphi$$

$$\eta_\alpha = \begin{bmatrix} \frac{1}{\sqrt{2}}\,\mathcal{F}_\alpha^{A\,\gamma}\,\Lambda_\gamma^A \\ -\frac{g^2}{4}\,\psi_\alpha\,\Lambda^{A\,\delta}\,\Lambda_\delta^A \\ -\frac{g^2}{4}\,\psi^*_{\mu\,\alpha}\,j_\Lambda^\mu \end{bmatrix} \tag{132}$$

$$\varphi = \frac{1}{2}\,\Lambda^{A\,\delta}\,\Lambda_\delta^A$$

Thus we find for B_α and $C = j\,\varphi$

$$B_\alpha = \begin{bmatrix} \left(\frac{1}{2\,g^2} - \frac{i}{16\,\pi^2}\,\Theta\right)\frac{1}{\sqrt{2}}\,\mathcal{F}_\alpha^{A\,\gamma}\,\Lambda_\gamma^A \\ -\left(\frac{1}{8} - \frac{i\,g^2}{16\,\pi^2}\,\Theta\right)\psi_\alpha\,\Lambda^{A\,\delta}\,\Lambda_\delta^A \\ -\left(\frac{1}{8} - \frac{i\,g^2}{16\,\pi^2}\,\Theta\right)\psi^*_{\mu\,\alpha}\,j_\Lambda^\mu \\ +\,\psi_\alpha\,\frac{1}{2}\,\Lambda^{A\,\delta}\,\Lambda_\delta^A \end{bmatrix} \tag{133}$$

$$C = \left(\frac{1}{2\,g^2} - \frac{i}{16\,\pi^2}\,\Theta\right)\frac{1}{2}\,\Lambda^{A\,\delta}\,\Lambda_\delta^A$$

results for B_α , C

Remark to D^{A} elimination [a]

It is possible, that the auxiliary fields D^{A} reappear through gauge fixing in a susy covariant way.

We can at any point reinsert them in $\mathcal{L} = (A , B_{\alpha} , C)$ using the quantities (eq. 120)

$$\widetilde{D}^{A} = D^{A} - \frac{g^{2}}{\sqrt{2}} \left(\psi^{\alpha} \Lambda_{\alpha}^{A} + \Lambda_{\dot{\beta}}^{*A} \psi^{*\dot{\beta}} \right) \tag{134}$$

through the substitution

$$\begin{aligned} \mathcal{L} &\rightarrow \mathcal{L} + \Delta \mathcal{L} \\ \Delta A &= j \tfrac{1}{2} \widetilde{D}^{A} \widetilde{D}^{A} \\ \Delta B_{\alpha} &= j \tfrac{1}{\sqrt{2}} \left[\widetilde{D}^{A} \Lambda_{\alpha}^{A} \right] \\ \Delta C &= 0 \end{aligned} \tag{135}$$

[a] Check all results in subsection 6c) .

7) The Legendre transform from (minimal) sources to classical fields representing the Lagrangean multiplet

We go back to the operator Lagrangean multiplet coupled to the minimal source extension as shown in eqs. (98 - 100) reproduced below

$$\underline{\Phi} \; = \; \left\{ \begin{array}{l} \vartheta^{\,2} \; \underline{H} \\ \\ + \; \vartheta^{\,\alpha} \; \underline{\eta}_{\;\alpha} \\ \\ + \; \underline{\varphi} \end{array} \right\} (\; x^{\,-} \;) \; \leftrightarrow$$

$$J \; = \; \left\{ \begin{array}{l} \vartheta^{\,2} \; (\; - \; m \;) \\ \\ + \; \vartheta^{\,\alpha} \; \psi_{\;\alpha} \\ \\ + \; j \end{array} \right\} (\; x^{\,-} \;) \tag{136}$$

$$j \; (\; y \;) \; = \; \tfrac{1}{2} \; \left[\; (\; 1 \; / \; g^{\,2} \; (\; y \;) \;) \; - \; \tfrac{i}{8 \pi^2} \; \Theta \; (\; y \;) \; \right]$$

$$\begin{aligned} \underline{L} \; (\; J \;) \; &= \; \int d^{\,2} \vartheta \; (\; \underline{\Phi} \; J \;) \; + \; h.c. \\ \underline{\mathcal{L}} \; (\; J \;) \; &= \; \underline{\Phi} \; J \; = \; (\; \underline{A} \; , \; \underline{B}_{\;\alpha} \; , \; \underline{C} \;) \end{aligned} \tag{137}$$

The explicit form of $\mathcal{L}$ is given in eqs. (130 and 133 - 135) and the general boundary conditions specified in eq. (102) .

In eqs. (136 - 137) underlined quantities are operator valued, distinguished in this section relative to the associated classical fields, defined below.

We do not discuss here gauge fixing and associated ghost action, which serve to define a gauge invariant measure in path space of the field configurations associated with the operator valued base variables

$$\begin{array}{l} \left\{ \, \underline{v} \, {}^{A}_{\mu} \, , \, \underline{\Lambda} \, {}^{A}_{\alpha} \, , \, \underline{\Lambda} \, {}^{* \; A}_{\dot{\alpha}} \, \right\} \longrightarrow \\ d \, \mu \, \left\{ \, v \, {}^{A}_{\mu} \, , \, \Lambda \, {}^{A}_{\alpha} \, , \, \Lambda \, {}^{* \; A}_{\dot{\alpha}} \, \right\} \end{array} \tag{138}$$

The measure μ in eq. (138) , subject to full renormalization, defines the generating functional

$$\begin{array}{l} Z \, (\, J \, , \, J^{\, *} \,) \; = \; \int d \, \mu \left\{ \, v \, {}^{A}_{\mu} \, , \, \Lambda \, {}^{A}_{\alpha} \, , \, \Lambda \, {}^{* \; A}_{\dot{\alpha}} \, \right\} \, \exp \; i \, S \\ S \; = \; \int d^{\, 4} y \, (\, \mathcal{L} \, (\, J \,) \, + \, \mathcal{L}^{\, *} \, (\, J^{\, *} \,) \,) \\ Z \, (\, J \, , \, J^{\, *} \,) \; = \; \exp \; i \, W \, (\, J \, , \, J^{\, *} \,) \end{array} \tag{139}$$

W defines the itransition from the operator valued Lagrangean multiplet (eq. 136) to its associated

classical fields

$$\underline{\Phi} = \left\{ \begin{array}{l} \vartheta^{2} \, \underline{H} \\ + \vartheta^{\alpha} \, \underline{\eta}_{\alpha} \\ + \underline{\varphi} \end{array} \right\} (y) \rightarrow$$

$$\rightarrow \Phi = \left\{ \begin{array}{l} \vartheta^{2} \, H \\ + \vartheta^{\alpha} \, \eta_{\alpha} \\ + \varphi \end{array} \right\} (y) \tag{140}$$

$$(\delta / \delta J (y)) \; W (J , J^{*}) = \Phi (y)$$

$$(\delta / \delta J^{*} (y)) \; W (J , J^{*}) = \Phi^{*} (y)$$

The effective potential Γ denotes the Legendre transforme of W

$$\Gamma (\Phi , \Phi^{*}) = \left\{ \begin{array}{c} \int d^{4} y \left[\begin{array}{c} \Phi (y) \, J (y) \\ + \Phi^{*} (y) \, J^{*} (y) \end{array} \right] \\ - W \end{array} \right\} \tag{141}$$

The arguments of Γ (eq. 141) are *the* classical fields pertaining to the operators $\underline{\Phi}$, $\underline{\Phi}^{*}$ as indicated in eq. (140) .

The former Φ , Φ^* are to be determined as functionals of the sources J , J^* from the defining, generating functional W, according to eq. (140) . The effective potential then determines associated sources, as multiple Legendre transforms are involutory [4]

$$
\begin{gathered}
(\delta / \delta \Phi (y)) \Gamma (\Phi , \Phi^*) = J (y) \\
(\delta / \delta \Phi^* (y)) \Gamma (\Phi , \Phi^*) = J^* (y) \\
W (J , J^*) = \left\{ \begin{array}{c} \int d^4 y \left[\begin{array}{c} \Phi (y) J (y) \\ + \Phi^* (y) J^* (y) \end{array} \right] \\ - \Gamma \end{array} \right\}
\end{gathered} \tag{142}
$$

Vanishing minimal sources, vanishing momenta and effective potential for the vacuum expected values of $\underline{\Phi}$, $\underline{\Phi}^*$

For vanishing minimal sources J , $J^* = 0$ eq. (142) requires an extremum (minimum) for the effective potential Γ

$$
\begin{gathered}
(\delta / \delta \Phi (y)) \Gamma (\Phi , \Phi^*) = 0 \\
(\delta / \delta \Phi^* (y)) \Gamma (\Phi , \Phi^*) = 0
\end{gathered} \tag{143}
$$

Remark on the naive classical limit $\hbar = 0$

If there were no renormalization to be done *also* in the classical limit, not to be confused with the classical fields Φ , Φ^* *within* a quantum field theory, the effective potential reduces to the negative classical action

$$\hbar \to 0 :$$

$$\Gamma (\Phi , \Phi^*) \to S_{cl} - S_{cl} (\Phi , \Phi^*) = 0$$

$$S_{cl} (\Phi , \Phi^*) = \int d^4 y \; L (\Phi , \Phi^*) \big|_{J = 0} (y) \tag{144}$$

The Lagrangean density $L (\Phi , \Phi^*) \big|_{J = 0}$ depends only on the limiting quantities J_∞ and can be constructed in this limit from A in eq. (130)

$$L (\Phi , \Phi^*) \big|_{J = 0} =$$

$$\left[\begin{array}{c} \frac{1}{g_\infty^2} \times \\ \times \Lambda^{*\,B}_{\dot\beta} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_\mu \delta_{BA} \\ - v_\mu^C \left(\mathcal{F}^C \right)_{BA} \end{array} \right\} \sigma^{\mu \, \dot\beta \alpha} \Lambda^A_\alpha \\ - \frac{1}{16 \pi^2} \Theta_\infty \partial_\mu j^\mu_\Lambda \\ - \frac{1}{4 g_\infty^2} F^{\mu\nu \, A} F^A_{\mu\nu} + \frac{1}{32 \pi^2} \Theta_\infty F^{\mu\nu \, A} \widetilde{F}^A_{\mu\nu} \\ - \frac{1}{2} m_\infty \Lambda^{A \, \alpha} \Lambda^A_\alpha + c.c. \end{array} \right] \tag{145}$$

Had we chosen sources for the primary fields, denoting them by

$\underline{\chi}_{\ primary} = \underline{\chi}$ **:** $\underline{\Lambda}^{\ A}_{\ \alpha}\ ,\ \underline{\Lambda}^{\ *\ A}_{\ \dot{\alpha}}$ **and** $\underline{v}^{\ A}_{\ \mu}$ **,**

the corresponding variations in the limit $\hbar \rightarrow 0$**, under the conditions that the variations** $\delta\ \chi$ **vanish at the boundaries of the action integral, the equivalent to eq. (143) would have reduced to the Euler-Lagrange field equations**

$$
\begin{aligned}
& (\ \delta\ /\ \delta\ \chi\ (\ y\ )\ )\ (\ -\ S_{\ cl}\ )\ (\ \chi\ ,\ \chi^{\ *}\ )\ =\ 0 \\
& (\ \delta\ /\ \delta\ \chi^{\ *}\ (\ y\ )\ )\ (\ -\ S_{\ cl}\ )\ (\ \chi\ ,\ \chi^{\ *}\ )\ =\ 0\ \rightarrow \\
& \partial_{\ \mu}\ L_{\ ,\ \partial_{\ \mu}\ \chi}\ -\ L_{\ ,\ \chi}\ =\ 0\ \textbf{and}\ \chi\ \rightarrow\ \chi^{\ *}
\end{aligned}
\tag{146}
$$

This shall illustrate the difference between minimal and primary source extension. [a]

Illustration : the free scalar field and primary source extension

We shall calculate for the primary source of a massive, free, complex scalar field $\underline{\chi}$ **the associated functionals**

[a] Prove the above derivation of the classical Euler-Lagrange equations in the classical limit.

$$L(J) \sim$$

$$-\underline{\chi}^{*}(m^{2}+\Box)\underline{\chi}+\left[J(y)\underline{\chi}+h.c.\right] \rightarrow$$

$$\underline{\chi}^{\prime}=\underline{\chi}+\Delta\chi$$

$$L=\left[\begin{array}{c} -\underline{\chi}^{\prime *}(m^{2}+\Box)\underline{\chi}^{\prime} \\ -\underline{\chi}^{\prime *}(m^{2}+\Box)\Delta\chi \\ -\{(m^{2}+\Box)\Delta\chi^{*}\}\underline{\chi}^{\prime} \\ +J(y)\underline{\chi}^{\prime}+\underline{\chi}^{\prime *}J^{*}(y) \\ +J(y)\Delta\chi+\Delta\chi^{*}J^{*}(y) \\ -\Delta\chi^{*}(m^{2}+\Box)\Delta\chi \end{array}\right] \tag{147}$$

We choose $\Delta\chi$, $\Delta\chi^{*}$ such as to cancel the source terms multiplying the operators $\underline{\chi}^{\prime}$, $\underline{\chi}^{\prime *}$ in eq. (147)

$$(m^{2}+\Box)\Delta\chi=J^{*}(y) \text{ and c.c.} \rightarrow$$

$$\Delta\chi=(m^{2}+\Box)^{-1}J^{*}(y) \text{ and c.c.} \tag{148}$$

$$\Box=\Box_{y}$$

Treating the Green function $(m^2 + \Box)^{-1}$ **as if it were real and well defined, we obtain for the generating functional** W

$$W(J, J^*) = \int d^4x\, d^4y\, J(x)\, G(x, y)\, J^*(y) \tag{149}$$
$$(m^2 + \Box)^{-1}_x . = \int d^4y\, G(x, y) .$$

Eq. (140) in this case becomes

$$(\delta/\delta J(y))\, W(J, J^*) = \chi(y) \,(= \chi_{cl})$$
$$\chi(y) = (m^2 + \Box)^{-1}_y J^* \rightarrow$$
$$J^* = (m^2 + \Box)\,\chi \text{ and c.c.} \tag{150}$$

and substituting into the corresponding eq. (141) we obtain

$$\Gamma(\chi, \chi^*) = \int d^4y\, \chi^*(m^2 + \Box)_y\, \chi = -S_{cl}(\chi, \chi^*) \tag{151}$$

But the situation changes in its interpretation for a constant nonvanisching source. $\rightarrow$

For a free field with a source $\lim_{y \to \infty} J(y) = J_{\infty}$ **we can deduce the potential energy (density-) term**

$$L(J) = L_{kin} - V(J))$$

$$V(J) \sim$$

$$m^2 \underline{\chi}^* \underline{\chi}(y) - J_{\infty} \underline{\chi}(y) - J^*_{\infty} \underline{\chi}^*(y) \rightarrow$$

$$\lim_{\mathcal{V} \to \infty} \mathcal{V}^{-1} \Gamma_{\mathcal{V}}(\chi, \chi^*) \rightarrow \gamma(\chi, \chi^*) =$$

$$m^2 \chi^* \chi(y) - J_{\infty} \chi(y) - J^*_{\infty} \chi^*(y) \tag{152}$$

In eq. (152) $\mathcal{V} = \mathcal{V}^4$ **denotes a four dimensional finitie volume, and the thermodynamic limit corresponds to** $\mathcal{V} \rightarrow \infty$**.**

The sourceless condition now implies in extension (not contradiction) of eqs. (143 and 151)

$$(\delta / \delta \chi(y)) \gamma(\chi, \chi^*) = 0 \ \textbf{and c.c.} \rightarrow$$

$$\chi(y) = \chi_{\infty} = J^*_{\infty} / m^2$$

$$\chi^*(y) = \chi^*_{\infty} = J_{\infty} / m^2 \tag{153}$$

In the result (eq. 153) the dangerous infrared limit $m \rightarrow 0$ **becomes apparent.**

We go back to section 3) (eqs. 12 , 13) and reproduce the structure of the full Lagrangean multiplet ($\mathcal{L} = (A , B_{\alpha} , C)$) given in eqs. (130, 133 - 135) below, in order to derive the infinitesimal susy transformations component by component

$$\mathcal{L} =$$

$$\left\{ \begin{array}{c} \vartheta^{2} \, \overline{\vartheta}^{2} \left(- \frac{1}{4} \Box C \right) \\ \vartheta^{2} \, \overline{\vartheta}_{\dot{\delta}} \left(\frac{-i}{2} \partial^{\dot{\delta}\alpha} B_{\alpha} \right) + \overline{\vartheta}^{2} \vartheta^{\alpha} \, 0 \\ \vartheta^{2} A + \vartheta^{\alpha} \sigma^{\mu}_{\alpha\dot{\beta}} \overline{\vartheta}^{\dot{\beta}} \left(\frac{-i}{2} \partial_{\mu} C \right) + \overline{\vartheta}^{2} \, 0 \\ + \overline{\vartheta}^{\dot{\beta}} \, 0 + \vartheta^{\alpha} B_{\alpha} \\ + C \end{array} \right\} \tag{154}$$

Also we recall the infinitesimal susy transformations (eqs. 30 , 33 $\rightarrow$)

$$\begin{aligned} q_{\alpha} &= (\partial_{\vartheta})_{\alpha} + \frac{i}{2} \overline{\vartheta}^{\dot{\beta}} (\partial_{x})_{\alpha\dot{\beta}} \\ \overline{q}_{\dot{\beta}} &= (\partial_{\overline{\vartheta}})_{\dot{\beta}} + \frac{i}{2} \vartheta^{\alpha} (\partial_{x})_{\alpha\dot{\beta}} \\ \overline{D}_{\dot{\beta}} &= - (\partial_{\overline{\vartheta}})_{\dot{\beta}} + \frac{i}{2} \vartheta^{\alpha} (\partial_{x})_{\alpha\dot{\beta}} \\ \overline{q}_{\dot{\beta}} &= 2 (\partial_{\overline{\vartheta}})_{\dot{\beta}} + \overline{D}_{\dot{\beta}} \end{aligned} \tag{155}$$

$$[\,\delta_U(\eta,\overline{\eta})\,]\,F(\vartheta,\overline{\vartheta},x)$$

$$\sim\; F_U(\vartheta,\overline{\vartheta},x) - F(\vartheta,\overline{\vartheta},x)$$

$$=\;\eta^{\alpha} q_{\alpha} F + \overline{\eta}^{\dot{\alpha}} \overline{q}_{\dot{\alpha}} F$$

$$q_{\alpha} = (\partial_{\vartheta})_{\alpha} + \tfrac{i}{2}\,\overline{\vartheta}^{\dot{\beta}}\,(\partial_x)_{\alpha\dot{\beta}} \tag{156}$$

$$\overline{q}_{\dot{\beta}} = (\partial_{\overline{\vartheta}})_{\dot{\beta}} + \tfrac{i}{2}\,\vartheta^{\alpha}\,(\partial_x)_{\alpha\dot{\beta}}$$

$$(\partial_x)_{\alpha\dot{\beta}} = (\partial_x)^{\mu}\,\sigma_{\mu\,\alpha\dot{\beta}}$$

$$(\partial_{\vartheta})_{\alpha} = \partial/\partial\vartheta^{\alpha}\;;\;(\partial_{\overline{\vartheta}})_{\dot{\beta}} = \partial/\partial\overline{\vartheta}^{\dot{\beta}}$$

Proceeding step by step we first calculate $(\partial_{\vartheta})_{\alpha}\,\mathcal{L}$ **and** $2\,(\partial_{\overline{\vartheta}})_{\dot{\beta}}\,\mathcal{L} = \overline{q}_{\dot{\beta}}\,\mathcal{L}$ **.**

$$(\partial_{\vartheta})_{\alpha}\,\mathcal{L} =$$

$$\left\{\begin{array}{c}
\vartheta^{2}\,\overline{\vartheta}^{2}\,0 \\
\vartheta^{2}\,\overline{\vartheta}_{\dot{\delta}}\,0 + \overline{\vartheta}^{2}\,\vartheta_{\alpha}\left(-\tfrac{1}{4}\,\Box\,C\right) \\
\vartheta^{2}\,0 + \vartheta_{\alpha}\,\sigma^{\mu\,\dot{\delta}\gamma}\,\overline{\vartheta}_{\dot{\delta}}\left(\tfrac{-i}{2}\,\partial_{\mu}\,B_{\gamma}\right) + \overline{\vartheta}^{2}\,0 \\
+\,\sigma^{\mu}_{\alpha\dot{\delta}}\,\overline{\vartheta}^{\dot{\delta}}\left(\tfrac{-i}{2}\,\partial_{\mu}\,C\right) + \vartheta_{\alpha}\,A \\
+\,B_{\alpha}
\end{array}\right\} \tag{157}$$

$$\overline{q}_{\dot{\beta}}\,\mathcal{L} = 2\,(\,\partial_{\overline{\vartheta}}\,)_{\dot{\beta}}\,\mathcal{L} =$$

$$\left\{\begin{array}{c} \vartheta^{2}\,\overline{\vartheta}^{2}\,0 \\ \vartheta^{2}\,\overline{\vartheta}_{\dot{\beta}}\left(\frac{1}{2}\,\Box\,C\right) + \overline{\vartheta}^{2}\,\vartheta^{\gamma}\,0 \\ \vartheta^{2}\,\left(-i\,\partial_{\gamma\dot{\beta}}\,B^{\gamma}\right) + \vartheta^{\gamma}\,\sigma^{\mu}_{\gamma\dot{\delta}}\,\overline{\vartheta}^{\dot{\delta}}\,0 + \overline{\vartheta}^{2}\,0 \\ +\,\overline{\vartheta}^{\dot{\beta}}\,0 + \vartheta^{\gamma}\,\sigma^{\mu}_{\gamma\dot{\beta}}\,(\,i\,\partial_{\mu}\,C\,) \\ +\,0 \end{array}\right\} \tag{158}$$

Next we compare $\overline{q}_{\dot{\beta}}\,\mathcal{L}$ with $\mathcal{L}$ (eq. 154) reproduced below

$$\mathcal{L} =$$

$$\left\{\begin{array}{c} \vartheta^{2}\,\overline{\vartheta}^{2}\left(-\,\frac{1}{4}\,\Box\,C\right) \\ \vartheta^{2}\,\overline{\vartheta}_{\dot{\delta}}\left(\frac{-i}{2}\,\partial^{\dot{\delta}\alpha}\,B_{\alpha}\right) + \overline{\vartheta}^{2}\,\vartheta^{\alpha}\,0 \\ \vartheta^{2}\,A + \vartheta^{\alpha}\,\sigma^{\mu}_{\alpha\dot{\beta}}\,\overline{\vartheta}^{\dot{\beta}}\left(\frac{-i}{2}\,\partial_{\mu}\,C\right) + \overline{\vartheta}^{2}\,0 \\ +\,\overline{\vartheta}^{\dot{\beta}}\,0 + \vartheta^{\alpha}\,B_{\alpha} \\ +\,C \end{array}\right\} \tag{159}$$

We multiply $\overline{q}_{\dot{\beta}}$ with an infinitesimal Grassmann spinor $\overline{\tau}^{\dot{\beta}}$ from the left and recast eq. (158) into the form

$$\delta_2 = \overline{\tau}^{\dot{\delta}} \, \overline{q}_{\dot{\beta}}$$

$$\delta_2 \mathcal{L} =$$

$$\left\{ \begin{array}{c} \vartheta^2 \, \overline{\vartheta}^2 \, 0 \\ \vartheta^2 \, \overline{\vartheta}_{\dot{\delta}} \, \overline{\tau}^{\dot{\delta}} \left(- \frac{1}{2} \, \square \, C \right) + \overline{\vartheta}^2 \, \vartheta^\gamma \, 0 \\ \vartheta^2 \, \overline{\tau}^{\dot{\delta}} \left(-i \, \partial_{\gamma \dot{\delta}} \, B^\gamma \right) + \vartheta^\gamma \, \sigma^\mu_{\gamma \dot{\delta}} \, \overline{\vartheta}^{\dot{\delta}} \, 0 + \overline{\vartheta}^2 \, 0 \\ + \overline{\vartheta}^{\dot{\beta}} \, 0 + \vartheta^\alpha \, \sigma^\mu_{\alpha \dot{\delta}} \, \overline{\tau}^{\dot{\delta}} \left(- i \, \partial_\mu \, C \right) \\ + 0 \end{array} \right\} \tag{160}$$

Comparing with eq. (159) we obtain, proceedeing slowly, term by term

$$\begin{aligned} \delta_2 \, C &= 0 \\ \delta_2 \left(\frac{-i}{2} \, \partial^{\dot{\delta}\alpha} \, B_\alpha \right) &= \overline{\tau}^{\dot{\delta}} \left(- \frac{1}{2} \, \square \, C \right) \rightarrow \\ \delta_2 \left(i \, \partial_\mu \, \sigma^{\mu \, \dot{\delta}\alpha} \, B_\alpha \right) &= \partial_\mu \left(\overline{\tau}^{\dot{\delta}} \, \partial^\mu \, C \right) \end{aligned} \tag{161}$$

We rewrite the last relation in eq. (161)

$$i\,\partial^{\dot\delta\alpha}\,\delta_2\,B_\alpha = \overline{\tau}^{\dot\delta}\,\Box\,C$$
$$\partial_{\gamma\dot\delta}\,\partial^{\dot\delta\alpha} = \delta_\gamma{}^\alpha\,\Box \tag{162}$$

Further we obtain

$$\delta_2\,A = \overline{\tau}_{\dot\delta}\left(-i\,\partial^{\dot\delta\alpha}\,B_\alpha\right)$$
$$\delta_2\,\partial_\mu\,C = 0 \leftarrow \delta_2\,C = 0 \tag{163}$$
$$\delta_2\,B_\alpha = \overline{\tau}^{\dot\delta}\,(-\,i\,\partial_{\alpha\dot\delta}\,C)$$

Finally – within the completion of δ_2 we operate with $i\,\partial^{\dot\gamma\,\alpha}$ on the last relation in eq. (163)

$$i\,\partial^{\dot\gamma\,\alpha}\,\delta_2\,B_\alpha = \overline{\tau}^{\dot\delta}\left(\partial^{\dot\gamma\,\alpha}\,\partial_{\alpha\dot\delta}\,C\right)$$
$$\rightarrow\ i\,\partial^{\dot\delta\,\alpha}\,\delta_2\,B_\alpha = \overline{\tau}^{\dot\delta}\,\Box\,C \tag{164}$$

7a) Susy transformations of the Lagrangean multiplet – δ_2

Thus we summarize results (eqs. 158 - 164)

$$\delta_2\,A = \overline{\tau}_{\dot\delta}\left(-i\,\partial^{\dot\delta\alpha}\,B_\alpha\right)$$
$$\delta_2\,B_\alpha = \overline{\tau}^{\dot\delta}\,(-\,i\,\partial_{\alpha\dot\delta}\,C) \tag{165}$$
$$\delta_2\,C = 0$$

7b) Susy transformations of the Lagrangean multiplet – $\delta_1 = \tau^{\delta} q_{\delta}$

We repeat from eq. (156) the form of q_{α}

$$q_{\alpha} = (\partial_{\vartheta})_{\alpha} + \frac{i}{2} \overline{\vartheta}^{\dot{\beta}} (\partial_x)_{\alpha\dot{\beta}} \tag{166}$$

The first part of $\delta_1 \mathcal{L}$, $(\partial_{\vartheta})_{\alpha} \mathcal{L}$ is displayed in eq. (157) .

We now turn to the second part –

$$\frac{i}{2} \overline{\vartheta}^{\dot{\beta}} \partial_{\alpha\dot{\beta}} \mathcal{L}$$

– remembering the form of $\mathcal{L}$ displayed in eq. (154) repeated for clarity below

$$\mathcal{L} =$$

$$\left\{ \begin{array}{c} \vartheta^2 \overline{\vartheta}^2 \left(- \frac{1}{4} \Box C \right) \\ \vartheta^2 \overline{\vartheta}_{\dot{\delta}} \left(\frac{-i}{2} \partial^{\dot{\delta}\alpha} B_{\alpha} \right) + \overline{\vartheta}^2 \vartheta^{\alpha} 0 \\ \vartheta^2 A + \vartheta^{\alpha} \sigma^{\mu}_{\alpha\dot{\beta}} \overline{\vartheta}^{\dot{\beta}} \left(\frac{-i}{2} \partial_{\mu} C \right) + \overline{\vartheta}^2 0 \\ + \overline{\vartheta}^{\dot{\beta}} 0 + \vartheta^{\alpha} B_{\alpha} \\ + C \end{array} \right\} \tag{167}$$

$$\frac{i}{2}\,\overline{\vartheta}^{\,\dot\beta}\,(\,\partial_{\,x}\,)_{\,\alpha\dot\beta}\,\mathcal{L} \;=$$

$$\left\{\begin{array}{c}
\vartheta^{\,2}\,\overline{\vartheta}^{\,2}\left(-\,\frac{1}{4}\,\Box\,B_{\,\alpha}\right) \\
\vartheta^{\,2}\,\overline{\vartheta}^{\,\dot\delta}\left(\frac{i}{2}\,\partial_{\,\alpha\dot\delta}\,A\right) + \overline{\vartheta}^{\,2}\,\vartheta_{\,\alpha}\left(\frac{1}{4}\,\Box\,C\right) \\
\vartheta^{\,2}\,0 \;+\; \vartheta^{\,\gamma}\,\overline{\vartheta}^{\,\dot\beta}\left(\frac{-i}{2}\,\partial_{\,\alpha\dot\beta}\,B_{\,\gamma}\right) + \overline{\vartheta}^{\,2}\,0 \\
+\,\overline{\vartheta}^{\,\dot\beta}\,\frac{i}{2}\,\partial_{\,\alpha\dot\beta}\,C \;+\; \vartheta^{\,\alpha}\,0 \\
+\,0
\end{array}\right\} \tag{168}$$

Next we repeat the first part $(\,\partial_{\,\vartheta}\,)_{\,\alpha}\,\mathcal{L}$ (eq. 157)

$$(\,\partial_{\,\vartheta}\,)_{\,\alpha}\,\mathcal{L} \;=$$

$$\left\{\begin{array}{c}
\vartheta^{\,2}\,\overline{\vartheta}^{\,2}\,0 \\
\vartheta^{\,2}\,\overline{\vartheta}_{\,\dot\delta}\,0 \;+\; \overline{\vartheta}^{\,2}\,\vartheta_{\,\alpha}\left(-\,\frac{1}{4}\,\Box\,C\right) \\
\vartheta^{\,2}\,0 \;+\; \vartheta_{\,\alpha}\,\sigma^{\,\mu\,\dot\delta\gamma}\,\overline{\vartheta}_{\,\dot\delta}\left(\frac{-i}{2}\,\partial_{\,\mu}\,B_{\,\gamma}\right) + \overline{\vartheta}^{\,2}\,0 \\
+\,\sigma^{\,\mu}_{\,\alpha\dot\delta}\,\overline{\vartheta}^{\,\dot\delta}\left(\frac{-i}{2}\,\partial_{\,\mu}\,C\right) + \vartheta_{\,\alpha}\,A \\
+\,B_{\,\alpha}
\end{array}\right\} \tag{169}$$

and add the two parts to obtain $q_{\,\alpha}\,\mathcal{L}$

$$q_\alpha \mathcal{L} =$$

$$\left\{ \begin{array}{c} \vartheta^2 \overline{\vartheta}^2 \left(-\frac{1}{4} \Box B_\alpha \right) \\ \vartheta^2 \overline{\vartheta}^{\dot\delta} \left(\frac{i}{2} \partial_{\alpha\dot\delta} A \right) + \overline{\vartheta}^2 \vartheta_\alpha 0 \\ \vartheta^2 0 + \vartheta^\gamma \overline{\vartheta}^{\dot\beta} \left(\begin{array}{c} \frac{-i}{2} \partial_{\alpha\dot\beta} B_\gamma \\ + \varepsilon_{\alpha\gamma} \frac{i}{2} \partial_{\kappa\dot\beta} B^\kappa \end{array} \right) + \overline{\vartheta}^2 0 \\ + \overline{\vartheta}^{\dot\beta} 0 + \vartheta_\alpha A \\ + B_\alpha \end{array} \right\} \tag{170}$$

We remark, as a check, that the chiral structure is indeed maintained. Finally we mulyiply with τ^α from the left to obtain $\delta_1 = \tau^\alpha q_\alpha$ (acting on $\mathcal{L} \rightarrow$)

$$\delta_1 \mathcal{L} = \tau^\alpha q_\alpha \mathcal{L} =$$

$$\left\{ \begin{array}{c} \vartheta^2 \overline{\vartheta}^2 \left(- \tau^\alpha \frac{1}{4} \Box B_\alpha \right) \\ \vartheta^2 \overline{\vartheta}^{\dot\delta} \left(- \tau^\alpha \frac{i}{2} \partial_{\alpha\dot\delta} A \right) + \overline{\vartheta}^2 \tau^\alpha \vartheta_\alpha 0 \\ \vartheta^2 0 + \vartheta^\gamma \overline{\vartheta}^{\dot\beta} \left(\begin{array}{c} - \tau^\alpha \frac{i}{2} \partial_{\alpha\dot\beta} B_\gamma \\ + \tau_\gamma \frac{i}{2} \partial_{\kappa\dot\beta} B^\kappa \end{array} \right) + \overline{\vartheta}^2 0 \\ + \overline{\vartheta}^{\dot\beta} 0 + \vartheta^\alpha \tau_\alpha A \\ + \tau^\alpha B_\alpha \end{array} \right\} \tag{171}$$

We compare with $\mathcal{L}$ (eq. 167) reproduced below

$$\mathcal{L} =$$

$$\left\{ \begin{array}{c} \vartheta^2 \overline{\vartheta}^2 \left(- \frac{1}{4} \Box C \right) \\ \vartheta^2 \overline{\vartheta}_{\dot\delta} \left(\frac{-i}{2} \partial^{\dot\delta\alpha} B_\alpha \right) + \overline{\vartheta}^2 \vartheta^\alpha 0 \\ \vartheta^2 A + \vartheta^\alpha \sigma^\mu_{\alpha\dot\beta} \overline{\vartheta}^{\dot\beta} \left(\frac{-i}{2} \partial_\mu C \right) + \overline{\vartheta}^2 0 \\ + \overline{\vartheta}^{\dot\beta} 0 + \vartheta^\alpha B_\alpha \\ + C \end{array} \right\} \tag{172}$$

We compare component by component

$$
\begin{aligned}
&\delta_1 \Box C = \tau^{\alpha} \Box B_{\alpha} \leftarrow \delta_1 C = \tau^{\alpha} B_{\alpha} \\
&\delta_1\, i\, \partial^{\dot{\delta}\alpha} B_{\alpha} = \tau_{\alpha}\, i\, \partial^{\dot{\delta}\alpha} A \\
&\uparrow \quad \delta_1 B_{\alpha} = \tau_{\alpha} A \\
&\delta_1 A = 0 \\
&\delta_1 \partial_{\gamma\dot{\beta}} C = \left(\begin{array}{c} \tau^{\alpha} \ \partial_{\alpha\dot{\beta}} B_{\gamma} \\ -\, \tau_{\gamma} \partial_{\kappa\dot{\beta}} B^{\kappa} \end{array} \right)
\end{aligned}
\tag{173}
$$

The last relation in eq. (173) can be transformed, using the identity

$$
\begin{aligned}
&\partial_{\alpha\dot{\beta}} B_{\gamma} - \partial_{\gamma\dot{\beta}} B_{\alpha} = \varepsilon_{\gamma\alpha} X \rightarrow \\
&\delta^{\gamma}_{\alpha} X = \partial_{\alpha\dot{\beta}} B^{\gamma} - \partial^{\gamma}_{\dot{\beta}} B_{\alpha} \rightarrow \\
&X = \partial_{\kappa\dot{\beta}} B^{\kappa}
\end{aligned}
\tag{174}
$$

From the identity (eq. 174) it follows [a]

$$
\begin{aligned}
&\delta_1 \partial_{\gamma\dot{\beta}} C = \partial_{\gamma\dot{\beta}} \tau^{\alpha} B_{\alpha} \\
&\uparrow \ \delta_1 C = \tau^{\alpha} B_{\alpha}
\end{aligned}
\tag{175}
$$

[a] Check all relations in eqs. (173 - 175) .

Results for $\delta_1 \mathcal{L}$

We summarize the susy variations $\delta_1 \mathcal{L}$ (eqs. 173 - 175) :

$$\begin{aligned} \delta_1 A &= 0 \\ \delta_1 B_\alpha &= \tau_\alpha A \\ \delta_1 C &= \tau^\alpha B_\alpha \end{aligned} \tag{176}$$

and repeat $\delta_2 \mathcal{L}$ (eq. 165) below

$$\begin{aligned} \delta_2 A &= \overline{\tau}_{\dot{\delta}} \left(-i\, \partial^{\dot{\delta}\alpha} B_\alpha \right) \\ \delta_2 B_\alpha &= \overline{\tau}^{\dot{\delta}} (- i\, \partial_{\alpha\dot{\delta}} C) \\ \delta_2 C &= \tau^\alpha B_\alpha \end{aligned} \tag{177}$$

7c) Susy transformations of the Lagrangean multiplet $- \delta = \delta_1 + \delta_2 = \tau^\gamma q_\gamma + \overline{\tau}^{\dot{\delta}} \overline{q}_{\dot{\delta}}$

Thus the full susy variations of $\mathcal{L}$ are

$$\delta = \delta (\tau , \overline{\tau})$$

$$\boxed{\begin{aligned} \delta A &= \qquad\qquad\qquad\quad \overline{\tau}^{\dot{\delta}} \left(-i\, \partial_{\gamma\dot{\delta}} B^\gamma \right) \\ \delta B_\alpha &= \tau^\gamma \varepsilon_{\alpha\gamma} A + \overline{\tau}^{\dot{\delta}} (- i\, \partial_{\alpha\dot{\delta}} C) \\ \delta C &= \tau^\gamma B_\gamma \end{aligned}} \tag{178}$$

8) Indirect derivation of the anomaly of the susy current

We return to the course of the Vienna seminar and the search for a *minimum* **of the effective action $\Gamma\ (\ \mathcal{L}\ ,\ \overline{\mathcal{L}}\ )$ (eqs. 136 - 143) , within the minimal source extension described in the previous section .**

Here we include the boundary values given in eq. (145)

$$\frac{1}{g\ ^2_{\ \infty}}\ ,\ \Theta\ _\infty\ ,\ m\ _\infty \tag{179}$$

in the definition of the arguments of Γ with the identification (eq. 136)

$$\begin{gathered}\mathcal{L}\ \rightarrow\ \Phi\ \leftrightarrow\ \Gamma\ (\ \mathcal{L}\ ,\ \overline{\mathcal{L}}\ )\ \rightarrow\ \Gamma\ (\ \Phi\ ,\ \overline{\Phi}\ )\\ \textbf{with}\quad \Phi\ \rightarrow\ \left\{\begin{array}{l}\vartheta\ ^2\ A\\ \\ +\ \vartheta\ ^\alpha\ B\ _\alpha\\ \\ +\ C\end{array}\right\}\ (\ x\ ^-\ )\end{gathered} \tag{180}$$

and – following ref. [5] – setting

$$\Theta\ _\infty\ \rightarrow\ \Theta\ _\infty\ -\ \Theta\ '\ \rightarrow\ 0 \tag{181}$$

We keep $m\ _\infty\ \rightarrow\ 0$ as an infrared regulater mass, to be set zero in the susy approaching limit.

The thermodynamic limit has been carefully described in ref. [3] and in the thesis of L. Bergamin [6] .

3 (of topics) On the road of conserved susy

We pursue the road along the logical possibility b_{+} **as defined in sections 1 and 2, hence assuming strict conservation of the susy (super)current**

$$j^{\dot{\gamma}}_{\ \beta\alpha} = k\,\Lambda^{*\,\dot{\gamma}\,A}\, f^{\ A}_{\{\beta\alpha\}}$$

$$j_{\mu\,\alpha} = (\sigma_{\mu})^{\ \beta}_{\dot{\gamma}}\; j^{\dot{\gamma}}_{\ \beta\alpha}$$

assuming as regula falsi : $\partial^{\mu}\, j_{\mu\,\alpha} = 0$

$$f^{\ A}_{\{\beta\alpha\}} = \frac{1}{2}\,(\sigma^{\mu\nu})_{\{\gamma\alpha\}}\,\mathcal{F}_{\mu\nu}$$

$$\mathcal{F}^{\ A}_{\mu\nu} = \partial_{\mu}\,\mathcal{W}^{\ A}_{\nu} - \partial_{\nu}\,\mathcal{W}^{\ A}_{\mu} + \mathcal{W}^{\ B}_{\mu}\,\mathcal{W}^{\ C}_{\nu}\; i\, f_{ABC}$$

$$\mathcal{F}^{\ A}_{\mu\nu} = \frac{1}{i}\,F^{\ A}_{\mu\nu}\ ;\ \mathcal{W}^{\ A}_{\mu} = i\, v^{\ A}_{\mu} \tag{182}$$

The numerical value of the constant k in eq. (182) , which ensures the normalisation condition in the equal time commutator in eq. (1) is here immaterial.

We recall the definitions in eqs. (74 - 76) →

$$(\sigma^{\mu\nu})_{\gamma}{}^{\delta} = \left(\varepsilon' \right)^{\delta\alpha} (\sigma^{\mu\nu})_{\{ \gamma\alpha \}}$$

$$(\sigma^{\mu})_{\gamma\dot{\beta}} (\sigma^{\nu})^{\dot{\beta}\delta} = \eta^{\mu\nu} \delta_{\gamma}{}^{\delta} + (\sigma^{\mu\nu})_{\gamma}{}^{\delta}$$

$$\eta^{\mu\nu} = diag (1 , -1 , -1 , -1)$$

$$\left(\sigma^{\mu\nu} = \left\{ \begin{array}{ll} \sigma_{r} & \textbf{for } {}^{\mu} = 0 , {}^{\nu} = r \\ \frac{1}{i} \varepsilon_{str} \sigma_{r} & \textbf{for } {}^{\mu} = s , {}^{\nu} = t \end{array} \right\} \right)_{\gamma}^{\ \delta} \tag{183}$$

This implies the symmetric property – with respect to the spinor indices $_{\beta\alpha}$ – of the current $j^{\dot{\gamma}}_{\ \beta\alpha}$.

Further the right chiral components project on the duality related combination (eq. 93 - 94) [a]

$$F^{+\ A}_{\mu\nu} = F^{\ A}_{\mu\nu} - i \widetilde{F}^{\ A}_{\mu\nu} \leftrightarrow \vec{B}^{\ A} - i \vec{E}^{\ A}$$

$$\widetilde{\sigma}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\varrho\tau} \sigma^{\varrho\tau}$$

$$\widetilde{\sigma}^{\mu\nu} = \left\{ \begin{array}{ll} i \sigma_{r} & \textbf{for } {}^{\mu} = 0 , {}^{\nu} = r \\ \varepsilon_{str} \sigma_{r} & \textbf{for } {}^{\mu} = s , {}^{\nu} = t \end{array} \right\} \tag{184}$$

$$\rightarrow \widetilde{\sigma}^{\mu\nu} = i \sigma^{\mu\nu}$$

---

[a] In the Euclidean transcription this becomes the anti-selfdual combination $\left(\vec{B}^{\ A} - \vec{E}^{\ A} \right)_{Eucl.}$ (show !) .

The logical deductions are shown step by step :

1 susy covariance

It follows from the assumption in eq. (182) that the effective potential $\Gamma \, (\, \Phi \, , \, \overline{\Phi} \,)$ in eq. (180) inherits full susy covariance.

2 Restriction of Φ to bosonic arguments – and eventual 'would be' goldstino condensates.

3 The effective potential must be bounded from below, yet exhibit a true minimum, corresponding to a (the) stable ground state.

4 The two bosonic arguments of $\Phi \ \leftrightarrow \ A \ , \ C$ characterizing the minimum correspond to vacuum expected values of the associated operators; a priori these values – any one of the two or both – can well be 0

5 Absence of spontaneous susy breaking requires $A \ = \ 0$, conversely $A \ \neq \ 0$ implies spontanous susy breaking and implies indeed the appearance of a Goldstone-fermion the goldstino.

6 The goldstino implies a positive vacuum energy density by the deduction in section 2.

7 The stability of the gauge boson condensate $\mathcal{B}^{\,2} \ = \ \langle \, \Omega \, | \ \frac{1}{4} \ F^{\ A}_{\ \mu\nu} \ F^{\ \mu\nu \ A} \ | \, \Omega \, \rangle$ requires a non-positive vacuum energy density

8 This apparently leaves only the logical possibility of no spontaneous breaking at all (b_0) – yet this is not given .

8a) Ad 1 , 2 : susy covariance and the fermionic arguments of Φ

From the discussion of the D^A eliminated Lagrangean multiplet in section 6) , 6a) (eqs. 98 - 135) we infer,

$$\begin{aligned} \Phi \ \rightarrow \ \Phi \ &= \ \Phi_{orig} \ J_{\infty} \ = \ \mathcal{L} \ \Big|_{\ J \ = \ 0} \\ \Phi \ &= \ (\ A \ , \ B_{\alpha} \ , \ C \ ) \end{aligned} \tag{185}$$

on the 'classical' fermionic fields B_{α} can involve – in the thermodynamic limit, to be discussed below – constant (in space-time) nontrivial associated condensates, for bilinears corresponding to B_{α} associated operators

$$\begin{aligned} & \underline{B}_{\alpha} \ (\ x \ ) \ \rightarrow \\ & \eta^{2} \ = \ \langle \ \Omega \ | \ : \ \underline{B}^{\alpha} \ \underline{B}_{\alpha} \ (\ x \ ) \ : \ | \ \Omega \ \rangle \ = \ constant \end{aligned} \tag{186}$$

The : : in eq. (186) denotes a normal ordering with respect to an unstable ground state.

The latter condensates (η^2 , $\overline{\eta}^2 = (\eta^2)^*$) , if not zero, do modify the minimum conditions. They must be distinguished from the condensates of the base fields $\Lambda^{A\,\alpha}\,\Lambda^{A}_{\alpha}$ and its hermitian conjugate.

The arguments of Γ can be reduced – in the 'static' or thermodynamic limit to the dependent chiral pair [1]

$$\begin{aligned} \Phi &= \Phi_0 = \left\{ \begin{array}{c} \vartheta^2 A \\ + \vartheta^\alpha B_\alpha \\ + C \end{array} \right\} (x_0) \\ \Psi &= \Phi_1 = \left\{ \begin{array}{c} \vartheta^2 0 \\ + \vartheta^\alpha 0 \\ + A^* \end{array} \right\} (x_0) \end{aligned} \tag{187}$$

The common and arbitrary space time argument x_0 for all classical fields is the result of a Taylor expansion of all quantities around this base point, whereby all derivatives (momenta) are set zero.

In this limit – which can well be dangerous because of infrared singularities – the static effectice potentials is constrained by susy covariance.

It is determined from a Kähler-like potential K and a complex conjugate pair of superpotentials S , S^* and takes the form

$$
\begin{aligned}
& K = K (\Phi , \Psi ; \Phi^* , \Psi^*) \\
& S = S (\Phi) , S^* = [S (\Phi)]^* \\
& b = \tfrac{1}{2} B^\alpha B_\alpha , \text{ and } b \to b^* \\
& \Gamma (\Phi , \Phi^*) = \\
& = \left[\begin{array}{c} \left(K_{1 \bar{1}} A^* A + S_1 A + S^*_{\bar{1}} A^* \right) \\ + \left(\begin{array}{c} - (K_{1 1 \bar{1}} A^* + S_{1 1}) b \\ - (K_{1 \bar{1} \bar{1}} A + S^*_{\bar{1} \bar{1}}) b^* \\ + K_{1 1 \bar{1} \bar{1}} b^* b \end{array} \right) \end{array} \right]
\end{aligned}
\tag{188}
$$

In eq. (188) the symbols $_1$, $_{\bar{1}}$ mean

$$
F_1 = (\partial / \partial C) F , \quad F_{\bar{1}} = (\partial / \partial C^*) F \tag{189}
$$

With the notation $K_{1 \bar{1}} = g$ and $S_1 = s$ we note the associated reduced arguments

$$
\begin{aligned}
g &= g (C , A^* ; C^* , A) \\
s &= s (C)
\end{aligned}
\tag{190}
$$

Γ **in eq. (188) then takes the form**

$$\Gamma(\Phi, \Phi^*) =$$

$$= \left[\begin{array}{c} (g A^* A + s A + s^* A^*) \\ + \begin{pmatrix} -(g_1 A^* + s_1) b \\ -(g_{\bar{1}} A + s^*_{\bar{1}}) b^* \\ + g_{1\bar{1}} b^* b \end{pmatrix} \end{array} \right] \tag{191}$$

$$g = g(C, A^*; C^*, A)$$

$$s = s(C), \; s^* = s^*(C^*)$$

It was proven by S. Portmann [7] that the restricted functional form

$$s = s(C, A^*) \to s(C) \tag{192}$$

is no losss of generality [a]

The quantities b , b^* **only occur in quadratic order as shown in eqs. (188 , 191) .**

It follows that Γ **is bounded from below exactly if**

$$\Gamma > -\infty \leftrightarrow g_{1\bar{1}} \geq 0 \tag{193}$$

[a] This should be checked when the quadratic fermionic quantities b , b^* are included .

8a 1) Ad 3 : Trivial (in)dependence of Γ on the fermionic arguments

The case where Γ is identically independent of the arguments b , b^* corresponds to the functional identities

$$\begin{aligned} g_1 \,,\ g_{\bar{1}} \ &\equiv\ 0 \ \rightarrow\ g \ =\ g\,(\,A\,,\ A^*\,) \\ s_1 \,,\ s^*_{\bar{1}} \ &\equiv\ 0 \ \rightarrow\ s\,,\ s^* \ =\ constant \end{aligned} \tag{194}$$

This is not acceptable, since as a consequence the effective potential minimum does not determine the spontaneous parameter

$$C \ \sim\ \langle\ \Omega\ |\ \Lambda^{A\gamma}\,\Lambda^{A}_{\gamma}\ |\ \Omega\ \rangle \tag{195}$$

rendering it a 'modulus'. We do not discuss this case further.

8a 2) Ad 3 : General dependence of Γ on the fermionic arguments

By (hermitian) bilinear completion and assuming $g\,,\ g_{1\bar{1}} \not\equiv\ 0$, Γ (eq. 191) takes the form

$$\Gamma (\Phi , \Phi^*) =$$

$$= \left[\begin{array}{c} g (A^* + \sigma^*) (A + \sigma) \\ - g \sigma^* \sigma \\ + \left(\begin{array}{c} g_{1 \bar{1}} (b^* - \tau^*) (b - \tau) \\ - g_{1 \bar{1}} \tau^* \tau \end{array} \right) \end{array} \right]$$

$$\sigma = s^* / g , \tau = (g_{\bar{1}} A + s^*_{\bar{1}}) / g_{1 \bar{1}}$$

$$\text{and } \sigma \to \sigma^* , \tau \to \tau^* \tag{196}$$

Thus as a first result we obtain the reduced effective potential

$$b |_{min} = \tau \text{ and c.c.} \to$$

$$\Gamma (\Phi , \Phi^*) \to$$

$$\to \left[\begin{array}{c} g (A^* + \sigma^*) (A + \sigma) \\ - g \sigma^* \sigma - g_{1 \bar{1}} \tau^* \tau \end{array} \right] \tag{197}$$

$$\sigma = s^* / g , \tau = (g_{\bar{1}} A + s^*_{\bar{1}}) / g_{1 \bar{1}}$$

$$\text{and } \sigma \to \sigma^* , \tau \to \tau^*$$

8b) Special values for the arguments of Γ

1) $A\ ,\ A^{*} = 0$

For $A\,(A^{*}) = 0$ **the effective potential becomes**

$$\Gamma\,|_{A=0} \to -\,|s_{1}|^{2}\,/\,\,g_{1\bar{1}}\,|_{A=0} \qquad (198)$$

As in section 8a 1) we discard the possibility that $s_{1}\ ,\ s_{\bar{1}}^{*}$ **are identically zero, for all of its arguments** $C\ ,\ C^{*}$ **respectively.**

It then follows that the overall minimum value of Γ **is negative. Since the effective potential is a highest component in its associated susy multiplet this fact in itself proves the spontaneous part of susy breaking.**

This is independent of the arguments $A\ ,\ A^{*}$ **for which this minimum is attained.**

The effective potential beeing bounded from below, a fortiori for restricted values of the arguments, now implies, remembering the condition $g_{1\bar{1}} > 0$ **(eq. 193)**

$$Max_{\,C\,,\,C^{*}}\left(|s_{1}|^{2}\,/\,\,g_{1\bar{1}}\,|_{A=0}\right) < \infty \qquad (199)$$

2) $\mid A \mid \;\rightarrow\; \infty$

For $\mid A \mid \;\rightarrow\; \infty$ **,** Γ **asymptotically becomes**

$$\Gamma \;\sim\; \Big(\mid A \mid^{2} \, / \, g_{\,1\,\bar{1}} \Big) \; \Big(g \; g_{\,1\,\bar{1}} \;-\; \mid g_{\,1} \mid^{2} \Big) \tag{200}$$

Also in the above limit the boundedness of Γ **from below imposes nontrivial conditions.**

9) Last but not least – the main problem : inconsistency of only sponteneous susy breaking

We return to the topics of the seminar in Vienna.

The identification of the gauge boson condensate :

The quantity $F \;=\; -\, g_{\,\infty}^{\,2} \; (\, A \;+\; A^{\,*} \,)$ **, related to the arguments** A , $A^{\,*}$ **of** Γ **for which** $\Gamma \;=\; \mathit{minimum}$ **is attained, is associated with the gauge boson condensate**

$$F \;\leftrightarrow\; \underline{F} \;=\; - \; \left(\begin{array}{c} \frac{1}{2} \, \underline{D}^{\,A} \, \underline{D}^{\,A} \; (a) \\ + \, \underline{\Lambda}^{\,*} \; i \; \sigma^{\,\mu} \; \overleftrightarrow{D}_{\,\mu} \; \underline{\Lambda} \; (b) \\ - \, \frac{1}{4} \, \underline{F}_{\,\mu\nu}^{\,A} \, \underline{F}^{\,A\,\mu\nu} \end{array} \right) \tag{201}$$

$$\rightarrow \qquad \langle \, \Omega \mid \underline{F} \mid \Omega \, \rangle$$

[a]

[a] (a) $\rightarrow$ 0 for vanishing sources, (b) : gauge invariant kinetic energy density for base fermions .

The quantities (a) and (b) in eq. (201) are thought not to contribute to the vacuum expected value $\langle \, \Omega \, | \, \underline{F} \, | \, \Omega \, \rangle$.

9a) Connection with trace anomaly (N=1 susy) and the sign of the vacuum energy density

The trace anomaly [8] retains its unique operator identity form specifically for the N=1 super Yang-Mills system, using the definitions in eq. (201)

$$\begin{aligned} \underline{\vartheta}\,_{\mu}^{\;\mu} \;&=\; (\, 2 \, \beta \, / \, g^{\,3} \,) \; \underline{F}\,_{g} \\ \underline{F}\,_{g} \;&=\; \tfrac{1}{4} \; \underline{F}\;\mu\nu^{\;A} \; \underline{F}^{\;A\;\mu\nu} \end{aligned} \tag{202}$$

The suffix in $\underline{F}\,_{g}$ as defined in eq. (202) indicates that the normalization of the operator $\underline{F}\,_{g}$ is implicitely dependent on the corresponding normalization of the running coupling constant $g \;=\; g \; (\; \mu \;)$ as discussed in [8] , while the energy momentum tensor components are – by definition – renormalization group invariant.

While eq. (202) is valid to all orders in the coupling constant, the lowest order perturbative contribution to the β function is

$$\begin{aligned} \beta \, / \, g^{\,3} \;&\sim\; b_{\,0} \, / \, (\, 16\pi^{\,2} \,) \\ b_{\,0} \;&=\; - \, 3 \, C_{\,2} \, (\, G \,) \end{aligned} \tag{203}$$

$C_2 (G)$ **in eq. (203) denotes the second Casimir operator of the underlying gauge group G, with the conventional** SUN **(embedding-) normalization**

$$C_2 (SUN) = N \tag{204}$$

This sets up the clash of signs for the vacuum expectation value

$$\begin{aligned} \langle \Omega \mid \underline{\vartheta}_{\mu\nu} \mid \Omega \rangle &= \varepsilon \, g_{\mu\nu} \\ \varepsilon &= (\beta / 2 g^3) \langle \Omega \mid \underline{F}_g \mid \Omega \rangle \end{aligned} \tag{205}$$

since the vacuum expected value of

$$\underline{F} = \frac{1}{2} \left(\left(\underline{\vec{B}}^A \right)^2 - \left(\underline{\vec{E}}^A \right)^2 \right) \tag{206}$$

requires the "Watt-less" positive sign for [9]

$$\begin{aligned} F_g = \langle \Omega \mid \underline{F}_g \mid \Omega \rangle \geq 0 \rightarrow \\ \varepsilon \leq 0 \end{aligned} \tag{207}$$

whereas the Goldstino situation (for $\varepsilon \neq 0$ **) requires according to section 2 (eq. 7)**

$$\varepsilon = | f_g |^2 \geq 0 \tag{208}$$

The clash of signs between eq. (207) and eq. (208) is the final result of this section [a]**.**

[a] We have come a long way ⋯

Superficially this might be interpreted as showing the correctness of no susy breaking of any kind (case $b_{\,0}$ in section 1) [10] , but really it constitutes an indirect proof, that the susy current must develop an anomalous divergence [11] .

10) Consequence and conjectured structure of the anomalous susy current divergence

a) Consequence

Assuming the existence of a genuine anomalous susy current the appearance of a massles goldstino mode is no more warranted, to the contrary the super Yang-Mills system develops – like QCD with one massless quark flavor – a mass gap.

The 'would be goldstino' mass is then generated similarly as the corresponding pseudoscalar $\eta^{\,\prime}$ like mass.

b) Conjectured anomaly structure

It was realized after the seminar beeing summarized here, that the susy current anomalous divergence may not be represented by a local operator, rather a local relation may involve a minimal (nonnegative) power p of the d'Alembert operator in the form

$$(\Box)^{p}\,\partial_{\mu}\,j^{\mu}_{\alpha}(x) = \delta^{(p)}_{\alpha}(x) \text{ and h.c.}$$
$$p = (0)\,,\ 1\,,\cdots \tag{209}$$

The local spinor operator $\delta^{(p)}_{\alpha}$ and the minimal power p in eq. (209) are to be considered as unknowns.

The naive structure of the susy current in spinor basis is (eq. 97)

$$j^{\dot\gamma}_{\beta\alpha} = k\,\Lambda^{*\,A\,\dot\gamma}\,\mathcal{F}^{A}_{\beta\alpha}\ ;\ dim\ j = \tfrac{7}{2}$$
$$j^{\mu}_{\alpha} = (\sigma^{\mu})^{\beta}_{\dot\gamma}\ j^{\dot\gamma}_{\beta\alpha} \tag{210}$$
$$\mathcal{F}^{A\,\gamma}_{\alpha} = \tfrac{1}{2}\,\mathcal{F}_{\mu\nu}\,(\sigma^{\mu\nu})_{\alpha}^{\ \gamma}$$

The normalization constant k in eq. (210) is here of no interest.

It follows for the dimension of $\delta^{(p)}_{\alpha}$

$$dim\ \delta^{(p)}_{\alpha} = (9 + 4\,p)\,/\,2 \tag{211}$$

While no candidate operators can be constructed for $p = 0$, for $p = 1$ the structure of $\delta^{(1)}_{\alpha}$ is

$$\begin{aligned} \delta_\alpha^{(1)} &\sim \Lambda^{*\,A}_{\dot\gamma}\, \Lambda^{*\,B\,\dot\gamma}\, \Lambda^{C\,\beta}\, \mathcal{F}^{\,D}_{\beta\alpha}\, c_{ABCD} \\ c_{ABCD} &= c_{BACD} \end{aligned} \tag{212}$$

In eq. (212) c_{ABCD} **denotes an invariant coupling coefficient for a product of 4 adjoint representations, with the symmetry constraint imposed by Fermi statistics.**

Simple choice are

$$\begin{aligned} (c_1)_{ABCD} &= \delta_{AB}\,\delta_{CD} \\ (c_2)_{ABCD} &= \delta_{AC}\,\delta_{BD} + \delta_{BC}\,\delta_{AD} \end{aligned} \tag{213}$$

We just note here the interesting combination of fields corresponding to c_1 **in eq. (213) , where the 'would be goldstino' field** g_α **is locally multiplied by the (anti) fermion bilinear** $\Lambda^*\,\Lambda^*$

$$\begin{aligned} \left(\delta_\alpha^{(1)}\right)_1 &\sim \left(\Lambda^{*\,A}_{\dot\gamma}\,\Lambda^{*\,A\,\dot\gamma}\right)\, g_\alpha \\ g_\alpha &= \Lambda^{B\,\beta}\,\mathcal{F}^{\,B}_{\beta\alpha} \end{aligned} \tag{214}$$

11) Outlook (conclusions)

The following conclusions are taken from the actual seminar. They do not reflect the entire body of deductions included here.

1) **The clear case consistent with all analogous derivations in QCD , for $N = 1$ super Yang-Mills systems exhibits the central property :**
   the susy curents $j_{\alpha}^{\ \mu}$, $j_{\dot{\alpha}}^{\ *\ \mu}$ are anomalous (in the generalized sense of eq. 209) .
   Precursor ideas have been defended by Aharony Casher [11] .
   Luzi Bergamin and Elisabeth Kraus were near .
   In addition susy is also spontaneously broken and develops a finite mass gap. As a consequence the 'would be goldstino' acquires a mass analogously to $\eta\,'$ in $N_{fl} = 1$ QCD.

2) **Let me express my gratitude and high esteem for my collaborators (since 1996) :**

Markus Leibundgut

Luzi Bergamin

Bernhard Scheuner †

He took his life a few months after this seminar

Samuel Portmann

3) Next ... $N = 4$?

Lets be open minded.

Thank you.

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